A

GEOMETRICAL TREATISE

OF

CONIC SECTIONS.

IN FOUR BOOKS.

TO WHICH IS ADDED,

A TREATISE ON THE PRIMARY PROPERTIES

OF

CONCHOIDS, THE CISSOID, THE QUADRATRIX,
CYCLOIDS, THE LOGARITHMIC CURVE,

AND THE

LOGARITHMIC, ARCHIMEDEAN, AND HYPERBOLIC SPIRALS.

BY THE

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OXFORD:

AT THE CLARENDON PRESS.

1802.

TO THE REVEREND

CYRIL JACKSON, D. D. F. R. S.

DEAN OF CHRIST CHURCH,

EQUALLY EMINENT FOR HIS OWN ABILITIES AND LEARNING,

AND FOR

HIS UNIFORM ENCOURAGEMENT AND PROMOTION

OF TALENTS AND ACQUIREMENTS IN OTHERS,

AS A

TESTIMONY OF THE HIGHEST ESTEEM FOR HIS CHARACTER,

AND AS A

TRIBUTE OF GRATITUDE FOR MANY IMPORTANT FAVOURS,

THIS WORK

IS MOST RESPECTFULLY INSCRIBED,

BY HIS MUCH OBLIGED AND FAITHFUL SERVANT,

THE AUTHOR.

ADVERTISEMENT.

THERE are two points of which it seems necessary that the reader of the following Treatise of Conic Sections should be apprized; first, what previous knowledge will be expected from him; and secondly, what extent of information the Treatise itself is intended to afford. The first of these will prevent the young student from entering upon the work till he is duly prepared, and the second will enable him to judge how far it is likely to contribute to the attainments which he has in view.

It is expected then, that the young student should understand thoroughly the first six Books of Euclid, the first twenty-one Propositions of the eleventh Book, the two first of the twelfth, and the first principles of Algebra, and Plane Trigonometry.

As no number can be assigned as a limit to the properties of the conic sections, any treatise on the subject can be supposed to contain only a selection of those which are most important and most useful, either generally, or with reference to the particular design of the Writer. In the present instance the design has been, to furnish the young Mathematician with

such a series of propositions as might prepare him for considering some of the most important truths in science, and enable him to enter on the study of natural philosophy, with the prospect of obtaining a thorough knowledge of the subject. According to these views the selection of properties and the extent of the work have been regulated; and at the same time the arrangement and division of the whole have been made with a design of accommodating two descriptions of readers. Those who are considered as constituting the first class are supposed to be desirous of a general but respectable portion of knowledge of the subject. For the use of such a perusal of the first three Books will be found sufficient, as they contain the properties of the sections most frequently referred to in pure and mixed mathematics. For those who rank under the second, or higher description, a knowledge of all the four Books will be requisite, as they complete the original design of rendering the whole a preparative for the Newtonian Philosophy. The Author flatters himself indeed, that he shall be found to have carried his elucidations of the Principia, in the present work, considerably beyond what have been attempted in other treatises of conic sections.

Something must now be added concerning the particular method, which has been adopted in these sheets, of deducing the primary properties of the sections from the nature of the cone.

It

*It is well known, that about the middle of the seventeenth century a difference of opinion took place among mathematicians concerning the proper ſource from which the properties of the conic ſections ſhould be deduced. But notwithſtanding the objections which then began to be made to their deduction from the cone, and which have ſince been continued, it appears to the Author of this work that the difficulties attributed to the deductions from it were not to be imputed to the ſolid itſelf, but that they were occaſioned ſolely by the manner in which the deductions had been made *. The early writers did not happen to perceive that the general and extenſive property, expreſſed in the thirteenth Propoſition of the firſt Book of this Treatiſe, could eaſily be obtained from the cone; and, not adverting to this, their deductions from the cone were ſometimes tedious and intricate.*

The above-mentioned property, as far as ſecants are concerned, occurs (I believe for the firſt time) in a folio volume, of which a treatiſe of conic ſections makes a part, entitled, Euclides Adauctus et Methodicus, &c. *publiſhed by Guarinus in* 1671. *The property to the ſame extent is to be found in Jones's* Synopſis Palmariorum Matheſeos, *publiſhed in* 1706; *but neither of theſe two authors conſidered the property as a fundamental one, nor do they ſeem*

* For foundations for ſyſtems, independent of the cone, ſee the Scholium in page 110, and the firſt ſeven articles in the Scholium at the end of the third Book.

to

to have been aware of the advantages it was capable of producing. Its extensive utility was first evinced in Hamilton's Conic Sections, published in Latin in 1758; *and on the appearance of this work objections to the cone ought to have ceased.*

This was my persuasion when I published my former treatise *, *and every deliberation on the subject since has tended to strengthen my conviction of its justice for the following reasons. First, the whole trouble with the cone is reduced to a very few demonstrations, for which no farther knowledge of Euclid is necessary than what is requisite for Spherical Trigonometry. Secondly, by this method the general properties are obtained with most ease and elegance. Lastly, by deducing the properties from the cone the treatise is rendered more extensively useful. No work on conic sections, confined to their description on a plane, can be applied to elucidations in Perspective, Projections of the Sphere, the Doctrine of Eclipses, and in some other particulars of the highest importance in science.*

For the rest it need only be said, that the manner in which the properties of the sections are classed and arranged, appeared to the Author, on the whole,

* In the year 1792 the author of the present work published a quarto volume, entitled, *Sectionum Conicarum Libri septem. Accedit Tractatus de Sectionibus Conicis, et de Scriptoribus qui earum doctrinam tradiderunt.* The last mentioned Tract contains a full historical account of the subject.

to be that which was beſt calculated to ſhew what properties are general, and what are appropriate to each of the ſections.

The treatiſe following that on conic ſections, in the preſent volume, contains only the moſt common properties of the curves ſpecified in the title page. It is intended as a preparative for thoſe who wiſh to inveſtigate the higher properties by means of Fluxions. In the firſt ſection methods of finding two mean proportionals and triſecting an angle, by means of conchoids, are inſerted. In the third ſection a method of dividing an angle in any given proportion, by means of the quadratrix, is given; as is alſo the quadrature of the circle, by means of the ſame curve.

N. B. When a demonſtration, in the following work, is effected by means of two ranks of magnitudes, which, taken two and two in the ſame or in a croſs order, have the ſame ratio to one another, they are placed thus,*

A : B : C : D

E : F : G : H;

A, B, C, D *repreſenting the firſt rank, and* E, F, G, H *the ſecond. Previous to this arrangement of the magnitudes, their ratio to one another is eſtabliſhed, and therefore it evidently appears in which of the two orders the magnitudes are proportional. In*

* That is either ex æquali, or ex æquali in proportione perturbata.

order

of contraction, $\left.\begin{array}{c}\text{t. A B}^2\\ \text{or}\\ \text{ſ. A B}^r\end{array}\right\}$, *which means the ſquare of* A B *if a tangent, or the rectangle under its ſegments if a ſecant.*

ERRATA.

Page 14. in the Corollary *read* can cut a ſcalene cone
— 128. line 4. from the bottom, *for* it is a ſecond diameter *read* C S is a ſecond diameter
— 226. line 9. *read* Nicomedes
— 240. line 21. *read* Dinoſtratus
— 247. line 7. *for* B D *read* E D
— 247. in the margin *read* Fig. 13.
— 258. line 16, &c. *read* analogous to a ſeries of logarithms, and the ordinates B F, C G, D H, &c. are analogous to the natural numbers of theſe logarithms.
— 259. line 15. and 18. *for* curve *read* ſpiral

LEMMAS

LEMMAS

FOR THE

FIRST THREE FOLLOWING BOOKS

OF

CONIC SECTIONS.

LEMMA I.

If the plane figure A F D *be bounded by the straight line* A D *and the curve* A F D, *and if the square of the straight line* F I, *drawn from any point* F *in the curve perpendicular to* A D, *be equal to the rectangle under the segments* A I, I D, *the figure will be a semicircle.* Fig. 1.

For let A D be bisected in C, and draw C F. Then (47. i.) the square of C F is equal to the squares of C I, I F together, and therefore, by hypothesis, equal to the square of C I together with the rectangle under A I, I D. But as A D is bisected in C, the square of A C or C D (5. ii.) is equal to the square of C I together with the rectangle under A I, I D. Consequently the square of C F is equal to the square of A C or C D; and therefore C F is equal to A C or C D. The figure A F D is therefore a semicircle. For Prop. IV. Book I.

If two straight lines be parallel, and a plane pass through each of them, the common section of these planes, if they cut one another, will be parallel to each of the parallel lines.

Fig. 2. Let two straight lines as A D, B C be parallel. Let the plane A D F pass through A D, and cut the plane B C F, passing through B C, in the straight line E F; the line of common section E F is parallel to A D, B C.

For let these planes be cut by two parallel planes E A B, F D C. Let the plane E A B cut the plane A B C D in the straight line A B, the plane A D F E in A E, and the plane B C F E in B E. Let the plane F D C cut the plane A B C D in the straight line D C, the plane A D F E in D F, and the plane B C F E in C F. Then the straight line A B is parallel to D C (16. xi.) A E is parallel to D F, and B E is parallel to C F. Hence (34. i.) A B, D C are equal; the angle E A B (10. xi.) is equal to the angle F D C, and the angle E B A is equal to the angle F C D. Consequently (26. i.) A E, D F are equal, and therefore (33. i.) A D, E F are equal and parallel. For the same reasons E F, B C are parallel. For Prop. VII. Book I.

LEMMA III.

If a straight line cut either of two parallel straight lines, and be in the same plane with them, it will, if sufficiently produced, cut the other.

Fig. 3. Let the straight lines A B, D E be parallel, and let C F be in the same plane with them, and cut A B in the point C; the straight line C F produced will cut D E.

For let any point G be taken in D E, and draw C G. Then as the angles B C G, E G C together (29. i.) are equal to two right angles, the angles F C G, E G C together

gether are lefs than two right angles, and therefore (ax. 12. i.) the ftraight lines C F, D E produced will meet one another. For Prop. VIII. Book I.

LEMMA IV.

If two ftraight lines cutting one another be parallel to a plane, a plane paffing through them will be parallel to the fame plane.

Let the two ftraight lines A B, C B, cutting one another in B, be parallel to the plane D G H E; the plane paffing through A B, C B is parallel to the plane D G H E. Fig. 4.

For let F be any point in the plane D G H E. Through A B and F let a plane be paffed, and let it cut the plane D G H E in the ftraight line D F H; and let a plane paffing through C B and F cut it in E F G. Then will A B be parallel to D F H, and C B will be parallel to E F G. For if not, then A B will meet D F H, and C B will meet E F G, and confequently A B, C B will meet the plane D G H E, in which D H, E G are, contrary to the hypothefis. The plane paffing through A B, C B (15.xi.) is therefore parallel to the plane D G H E. For Prop. X. Book I.

LEMMA V.

If the firft of eight ftraight lines be to the fecond as the third to the fourth, and if the fifth be to the fixth as the feventh to the eighth; then the rectangle under the firft and fifth will be to the rectangle under the fecond and fixth as the rectangle under the third and feventh to the rectangle under the fourth and eighth. And if the rectangle under the firft and fifth, of eight ftraight lines, be to the rectangle under the fecond and fixth as the rectangle under the third and feventh to the rectangle under the fourth and eighth, and if the firft be to the fecond as

 the

the third to the fourth, then the fifth will be to the sixth as the seventh to the eighth.

Fig. 5. Part I. Let A B, the first of eight straight lines, be to B C the second, as D E the third to E F the fourth, and let G B the fifth be to B H the sixth as I E the seventh to E K the eighth; then the rectangle under A B, G B is to the rectangle under B C, B H as the rectangle under D E, I E to the rectangle under E F, E K.

For let A B, B C be in a straight line; G B, B H be in a straight line; D E, E F be in a straight line; and I E, E K be in a straight line; and let these straight lines be at right angles to one another, and let the rectangles be completed as represented in the figure. Then A G is the rectangle under A B, G B; C H is the rectangle under B C, B H; D I is the rectangle under D E, I E; and F K is the rectangle under E F, E K. By hypothesis A B : B C : : D E : E F, and therefore (II. v. and I. vi.) A G : G C : : D I : I F. Again, by hypothesis, G B : B H : : I E : E K, and therefore (II. v. and I. vi.) G C : C H : : I F : F K. Consequently,

A G : G C : C H
D I : I F : F K,

and therefore (22. v.) A G : C H : : D I : F K.

Fig. 5. Part II. The construction, with respect to the rectangles, remaining as stated above, let the rectangle A G be to the rectangle C H as the rectangle D I to the rectangle F K, and let A B be to B C as D E to E F; then G B is to B H as I E to E K.

For, by hypothesis and inversion, B C : A B : : E F : D E; and therefore (II. v. and I. vi.) G C : A G : : I F : D I. Again by hypothesis A G : C H : : D I : F K. Consequently,

G C : A G : C H
I F : D I : F K;

and

and therefore (22. v.) G C : C H : : I F : F K. But (1. vi.) G C : C H : : G B : B H, and I F : F K : : I E : E K; and therefore (11. v.) G B : B H : : I E : E K. For Prop. X. Book I.

LEMMA VI.

If the points C, D *be so situated in the straight line* A B, *that the rectangle* D A C *is equal to the rectangle* C B D, *then* A C *is equal to* B D: *or if the rectangle* A C B *be equal to the rectangle* B D A, *then* A C *is equal to* B D. Fig. 6.

Case I. Let C D be bisected in E, and then (6. ii.) the rectangle D A C together with the square of E C is equal to the square of A E; and the rectangle C B D together with the square of E D is equal to the square of B E. The squares of A E, B E are therefore equal, and consequently A E is equal to B E, and A C is equal to B D.

Case II. Let A B be bisected in E, and then (5. ii.) the rectangle A C B together with the square of E C is equal to the square of A E or E B; and the rectangle B D A together with the square of E D is equal to the square of E B. The squares of E C, E D are therefore equal, and consequently E C is equal to E D, and A C is equal to B D. For Prop. I. Book II.

LEMMA VII.

If a straight line touch a circle, and two straight lines cutting the circle pass through the point of contact, and meet a straight line parallel to the tangent, the rectangle under the segments of the one, between the point of contact and circumference, and between the point of contact and straight line parallel to the tangent, will be equal to the rectangle under the segments of the other, between the point of contact and circumference and between the point of contact and straight line parallel to the tangent.

 Let

Fig. 7. and 8. Let the ſtraight line A B be parallel to the ſtraight line R G touching the circle E P F in the point P, and let the two ſtraight lines E P, F P, paſſing through P, meet the circumference again, the one in E and the other in F, and let E P meet the ſtraight line A B in C, and F P meet it in D; then the rectangle under E P, P C is equal to the rectangle under F P, P D.

For E F being drawn, the angle E F P (32. iii.) is equal to the angle R P E, which (29. i.) is equal to the angle P C D. The triangles E P F, D P C are therefore equiangular, and (4. vi.) E P : P F : : D P : P C. Conſequently, (16. vi.) the rectangle under E P, P C is equal to the rectangle under F P, P D*. For Prop. I. Book III.

Cor. 1. If A B cut the circle in B, and P B be drawn, it may be proved in the ſame way that the rectangle under F P, P D is equal to the ſquare of P B. For the tangent R P being produced to G, and B F being drawn, the angle B P G (32. iii.) is equal to the angle B F P; and (29. i.) it is alſo equal to the angle D B P. The triangles B F P, D B P are therefore equiangular, and (4. vi.) F P : P B : : P B : P D, and (17. vi.) the rectangle under F P, P D is equal to the ſquare of P B.

Fig. 8. *Cor.* 2. The reſt remaining as above, if B G be drawn parallel to F P, B G is (34. i.) equal to P D, and therefore by the above $FP = \frac{PB^2}{BG}$.

LEMMA VIII.

If the firſt of three ſtraight lines be to the third as the ſquare of the ſum of the firſt and ſecond to the ſquare of the ſum of the ſecond and third, the ſecond will be a mean

* The ſtraight line A B may be on either ſide of the tangent R G, and it is not neceſſary, upon being produced indefinitely, that it ſhould meet the circumference of the circle.

first and second to the square of the difference of the se-cond and third, the second will be a mean proportional between the first and third.

Part I. Let A denote the first, B the second, and C the third, of the straight lines. Then by hypothesis $A : C :: \overline{A+B}^2 : \overline{B+C}^2$; and it is to be proved that B is a mean proportional between A and C.

Let D be a mean proportional between A and C, and then, by inversion, $D : A :: C : D$, and (18. v.) $A+D : A :: D+C : D$; and therefore by alternation $A : D :: A+D : D+C$. Consequently (22. vi.) $A^2 : D^2 :: \overline{A+D}^2 : \overline{D+C}^2$. But (Cor. 2. 20. vi.) $A : C :: A^2 : D^2$, and therefore by hypothesis (and 11. v.) $\overline{A+B}^2 : \overline{B+C}^2 :: \overline{A+D}^2 : \overline{D+C}^2$. Consequently (22. vi.) $A+B : B+C :: A+D : D+C$; and therefore, by conversion, $A+B : A-C :: A+D : A-C$, and (14. v.) $A+B$ is equal to $A+D$. Consequently B is equal to D, and therefore B is a mean proportional between A and C.

Part II. Let A denote the first of the three straight lines, B the second, and C the third. Then by hypothesis $A : C :: \overline{A-B}^2 : \overline{B-C}^2$; and it is to be proved that B is a mean proportional between A and C.

Let D be a mean proportional between A and C. Then $A : D :: D : C$, and by conversion $A : A-D :: D : D-C$; and by alternation, $A : D :: A-D : D-C$. Consequently (22. vi.) $A^2 : D^2 :: \overline{A-D}^2 : \overline{D-C}^2$. But (Cor. 2. 20. vi.) $A : C :: A^2 : D^2$; and therefore by hypothesis (and 11. v.) $\overline{A-B}^2 : \overline{B-C}^2 :: \overline{A-D}^2 : \overline{D-C}^2$. Consequently (22. vi.) $A-B : B-C :: A-D : D-C$; and therefore (18. v.) $A-C : B-C :: A-C : D-C$. Hence (14. v.) $B-C$ is equal to

 D — C,

D − C, and therefore B is equal to D. Confequently B is a mean proportional between A and C. For Prop. XXIII. Book III.

A GEOMETRICAL TREATISE OF CONIC SECTIONS.

BOOK I.

Containing general Properties deduced from the Cone.

DEFINITIONS.

I.

IF through the point V, without the plane of the circle AFB, a ſtraight line AVD be drawn, and extended indefinitely both ways, and if the point V remain fixed, and the ſtraight line AVD be moved round the whole circumference of the circle, two Superficies will be generated by its motion, each of which is called a *Conical Superficies*; and theſe mentioned together are called *Oppoſite Superficies*. Fig. 9.

Cor. A ſtraight line drawn from the fixed point V to any point G in either ſuperficies is wholly in that ſuperficies; and, being produced, the part on the other ſide of V is wholly in the oppoſite ſuperficies. For a ſtraight line

superficies; and, being produced, the part beyond V is in the opposite superficies. Hence the Cor. is evident; for only one straight line can be drawn from V to G, as two straight lines cannot inclose a space.

II.

The solid contained by the conical superficies and the circle A F B is called *a Cone.*

III.

The fixed point V is called the *Vertex of the Cone.*

IV.

The circle A F B is called the *Base of the Cone.*

V.

Any straight line drawn through the vertex of the cone to the circumference of the base is called a *Side of the Cone.*

VI.

A straight line V C, drawn through the vertex of the cone and the center of the base, is called the *Axis of the Cone.*

VII.

If the axis of the cone be perpendicular to the base, it is called a *Right Cone.*

VIII.

If the axis of the cone be not perpendicular to the base, it is called a *Scalene Cone.*

IX.

A plane is said to *touch a conical superficies*, when it meets the superficies, and when, being produced indefinitely, in any direction, it falls without the superficies.

X.

A straight line which meets a conical superficies, and which, being produced both ways, falls without the super-

superficies, is called a *Tangent*; but a straight line which meets a conical superficies in two points, or each of the opposite superficies in one point, is called a *Secant*. BOOK I.

XI.

A straight line is said to be parallel to a plane, when both being produced ever so far, both ways, they do not meet.

XII.

If a cone be cut by a plane, their common intersection is called a *Conic Section*.

XIII.

The common intersection of any plane, not passing through the vertex of the cone, with the conical superficies, is called the *Curve of a Conic Section*.

PROP. I.

If a cone be cut by a plane passing through the vertex, the section will be a triangle.

Let the cone V A F B be cut by a plane passing through V the vertex, and let V A B be the common intersection of the cone and plane; the section V A B is a triangle. Fig. 9.

For let the plane, passing through V, cut the plane of the base in the straight line (3. xi.) A B, and the circumference of the base in the points A, B; and let the straight lines V A, V B be drawn. Then, as the points V, A, B are in the plane cutting the cone, the straight lines V A, V B are wholly in the same plane; and as the points A, B are in the conical superficies, the straight lines V A, V B are also wholly in the superficies, by the corollary to the first Definition. The straight lines V A, V B are therefore the common intersections of the conical superficies and the plane cutting the cone; and consequently the section V A B is a triangle.

Cor.

BOOK I.

Cor. If a plane, passing through the vertex, cut a cone, it will cut the opposite superficies in two straight lines, and only in those two. For if the plane V A B be extended on both sides of the vertex, it will cut the opposite superficies in the straight lines V A, V B produced, and in them only. This is evident from the above, and the corollary to the first Definition.

PROP. II.

If either of the opposite conical superficies be cut by a plane parallel to the base of the cone, the common interfection of the superficies and the plane will be the circumference of a circle, and its center will be in the axis of the cone.

Fig. 10. Let the superficies V A B D, or its opposite superficies, be cut by a plane parallel to the base A B D of the cone, and let F G H be the common interfection of the superficies and this plane; F G H is the circumference of a circle, and its center is in V C, the axis of the cone, or in V C produced.

Let C be the center of the base, and let the axis V C cut the plane F G H in the point I. From the point I, and in the plane F G H, draw any two straight lines I F, I G to the conical superficies. Through V I C, I F let a plane be passed, and let it cut the superficies in the side V F A, and the base of the cone in the straight line C A. Let a plane also be passed through V I C, I G, and let it cut the superficies in the side V G B, and the base of the cone in the straight line C B. Then (16. xi.) F I, A C, and also G I, B C are parallel to one another, each to each: and (29. i.) the triangles A C V, F I V, and also the triangles B C V, G I V are equiangular, each to each. Consequently (4. vi.) A C : F I :: V C : V I, and also V C : V I :: B C : G I; and therefore (11. v.) A C : F I :: B C : G I, and as A C is equal to B C, F I is (14. v.) equal to G I. In the same manner it may be proved that

e.

Plate

Fig. 1.

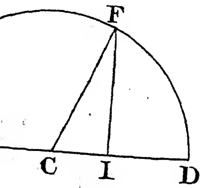


Fig. 2.

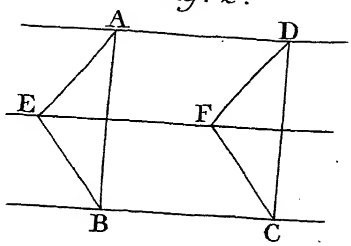


Fig. 3.

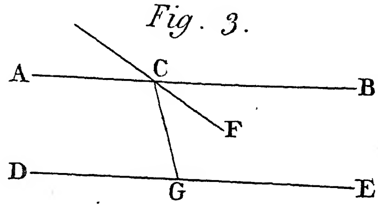


Fig. 4.

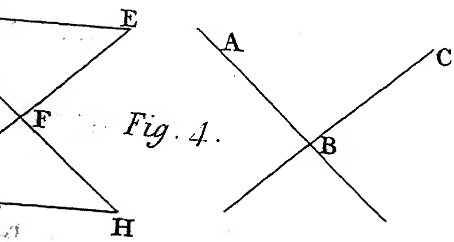


Fig. 5.

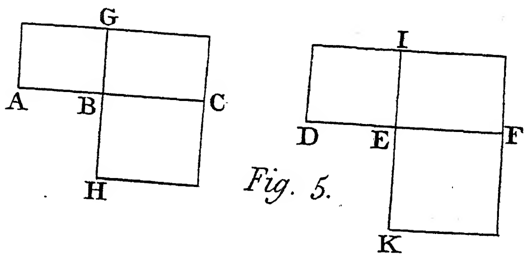


Fig. 6.

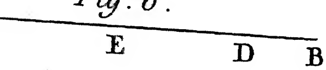


Fig. 7.

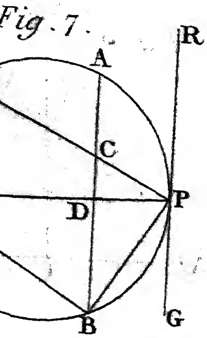


Fig. 8.

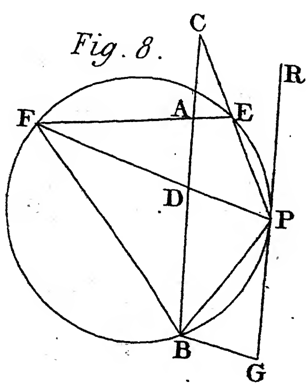
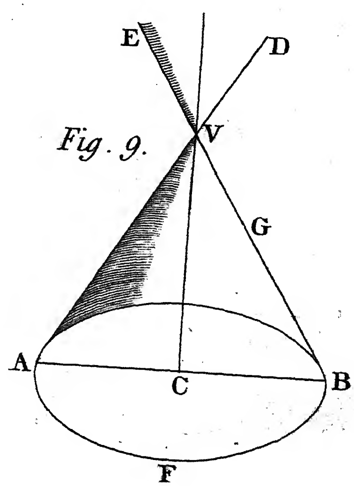


Fig. 9.



that any other ſtraight line drawn in the plane F G H from the point I to the conical ſuperficies is equal to F I; and therefore F G H is the circumference of a circle, and the point I, in the axis V C, is its center.

Cor. 1. From this Propoſition, and the firſt Definition, it appears that any circle parallel to the baſe of the cone, having its center in the axis, and its circumference in either of the oppoſite ſuperficies, may be taken for the baſe of the cone.

Cor. 2. The ſolid contained by the conical ſuperficies V F G H, oppoſite to V A B D, and the circle F G H, is a cone.

PROP. III.

If a ſcalene cone be cut through the axis by a plane perpendicular to the baſe, of the ſides of the ſection, meeting in the vertex, one will be the greateſt, and the other the leaſt of all the ſides of the cone.

For let V N O P be a ſcalene cone, of which V is the vertex, N O P the baſe, and C the centre of the baſe. Let the ſtraight line V B be perpendicular to the plane of the baſe, and meet it in B. Draw the ſtraight line B C, and let it meet the circumference of the baſe in the points P, N. Through the ſtraight lines V B, B C let a plane be paſſed, and let it cut the cone; and let the ſection formed, with the cone, be the triangle V N P, as in the firſt Propoſition. Then as the plane of the triangle V N P paſſes through C, it cuts the cone through the axis*; and, as it paſſes through V B, it is alſo (18. xi.) perpendicular to the Fig. 11. and 12.

* As the repreſentation of the axis could not render the demonſtration more perſpicuous, it was intentionally omitted in the figures. The Reader will find the ſame omiſſion in the figure for Prop. IV. and V. and intentionally made, for the ſame reaſon.

baſe

BOOK I.

base N O P. If therefore the point B be farther from N than from P, it remains to be proved, that V N is greater and V P less than any other side of the cone.

Let V O be any other side of the cone, and let it meet the circumference of the base in O, and draw B O. Then, as V B is perpendicular to the plane of the base, the angles V B N, V B P, V B O are right angles; and therefore (47. i.) the square of V N is equal to the squares of V B, B N together; and the square of V O is equal to the squares of V B, B O together; and the square of V P is equal to the squares of V B, B P together. But (7. and 8. iii.) B N is greater than B O, and B O is greater than B P; and therefore the square of B N is greater than the square of B O, and the square of B O is greater than the square of B P. Consequently the square of V N is greater than the square of V O, and the square of V O is greater than the square of V P. Of all the sides of the cone therefore, V N is the greatest, and V P is the least.

If the perpendicular V B fall into the circumference of the base, then B and P will coincide; and (referring to 15. iii. instead of 7. and 8. iii) the demonstration will be the same as above*.

Cor. As there can be only one perpendicular to a plane (13. xi.) drawn from the same point above the plane, it is evident from the demonstration of this Proposition that only one plane can cut a cone through the axis, and be perpendicular to the base.

* It is evident that all the sides of a right cone are equal to one another. For, in this case, the perpendicular to the base, drawn from V, will fall into C, the centre, according to the seventh definition.

PROP.

BOOK I.

PROP. IV.

Fig. 13.

Let the ſcalene cone V N O P *be cut by a plane paſſing through the axis, and perpendicular to the baſe* N O P, *and let the common ſection be the triangle* V N P; *in the ſide* V P *take any point* D, *and in the plane of the triangle make the angle* V D A *equal to the angle* V N P; *then if the cone be cut by a plane paſſing through* D A, *and perpendicular to the triangle* V N P, *its common ſection* A F D B *with the cone will be a circle.*

For let the ſide V P be leſs than the ſide V N, as in the preceding Propoſition, and produce A D to T and N P to R. Then, as V N is greater than V P, the angle V P N (18. i.) is greater than the angle V N P. But the angles V N P, V D A are equal, by hypotheſis; and as the angle P D T is equal (15. i.) to the angle V D A, the angle V P N is greater than the angle P D T. The angles V P N, D P R together are therefore greater than the angles P D T, D P R together; and conſequently the angles P D T, D P R together are leſs than two right angles. If therefore the ſtraight lines A D, N P be ſufficiently produced they will meet. Let them be produced and meet in R; and let the plane of the ſection A F D B cut the plane of the baſe N O P in the ſtraight line R S. In D A take any point I, and let the cone be cut by a plane paſſing through I and parallel to the baſe; and let the ſection formed be the circle H F K B, as in the ſecond Propoſition. Let the circle H F K B cut the triangle V N P in the ſtraight line H I K, and the ſection A F D B in the ſtraight line F I B. Then as the ſection H F K B is parallel to the plane of the baſe, and as theſe parallel planes are cut by the plane of the ſection A F D B, the common ſections (16. xi.) F I B, S R are parallel; and as the plane of the ſection A F D B, and the plane of the baſe N O P are perpendicular to the plane

BOOK I. plane of the triangle V N P, and cut one another in S R, the ſtraight line S R (19. xi.) is perpendicular to the plane of the triangle V N P. The ſtraight line F I B (8. xi.) is therefore perpendicular to the plane V N P, and conſequently (4. xi.) perpendicular to H K, A D. But, as the plane of the triangle V N P paſſes through the axis, H K is a diameter of the circle H F K B by the ſecond Propoſition, and therefore (3. iii.) F B is biſected in I. Conſequently (35. iii.) the rectangle under K I, I H is equal to the ſquare of F I or B I. Again, as the circle H F K B is parallel to the plane of the baſe, and as theſe parallel planes are cut by the triangle V N P, the common ſections (16. xi.) H I K, N P are parallel. The angle A H I (29. i.) is therefore equal to the angle V N P, and conſequently equal to the angle K D I. The angles (15. i.) A I H, K I D are alſo equal, and therefore the triangles A I H, K I D are equiangular. Conſequently (4. vi.) A I : I H : : K I : I D, and (16. vi.) the rectangle under A I, I D is equal to the rectangle under K I, I H; and therefore, by the above, the rectangle under A I, I D is equal to the ſquare of F I or I B. The ſection A F D B is therefore a circle, by the firſt Lemma.

The circle A F D B formed in a ſcalene cone, in the manner mentioned in the Propoſition, is called *a Subcontrary Section.*

PROP. V.

If a conic ſection be a circle, and be not parallel to the baſe of the cone, it will be a ſubcontrary ſection.

Fig 13. Let the cone V N O P be cut by a plane not parallel to the baſe N O P, and let the ſection A F D B formed by it, with the cone, be a circle; A F D B is a ſubcontrary ſection.

For let I be the point in which the axis of the cone meets

meets the circle A F D B, and through I let a plane be paſſed parallel to the baſe, and let F K B H be the circle formed by it with the cone, as in the ſecond Propoſition. Let B F be the common ſection of this circle with the circle A F D B. Then by Prop. II. the point I is the center of the circle F K B H, and conſequently B F is biſected in I. Through I draw in the circle A F D B the ſtraight line A D at right angles to B F; and through A D and V, the vertex, let a plane be paſſed, and let V N P be the triangle formed by it with the cone, as in the firſt Propoſition. Let H K be the line of common ſection of the triangle V N P and the circle F K B H. Let L be any point in A D, and through L let a plane be paſſed parallel to the baſe N O P, or to the circle F K B H. Let M C E G be the circle formed by this plane with the cone, and let M E be its line of common ſection with the triangle V N P, and C L G its line of common ſection with the circle A F D B. Then (16. xi.) the ſtraight lines H K, M E are parallel, as are alſo B I F, C L G; and as A I B is a right angle, A L C is (29. i.) a right angle. Again, as the ſtraight line A D biſects the ſtraight line B F at right angles, A D (Cor. 1. iii.) is the diameter of the circle A F D B. The ſtraight line G C (3. iii.) is therefore biſected in L. But as I is the point in which the axis of the cone meets the circle A F D B, it is evident that the triangle V N P cuts the cone through the axis, and conſequently by Prop. II. M E is a diameter of the circle M C E G, and the point L is not its center. Hence the diameter M E (3. iii.) biſects G C in L at right angles, and G L is at right angles to A D, M E, and therefore it is at right angles to the triangle (4. xi.) V N P. Conſequently (18. xi.) each of the ſections A F D B, M G E C is at right angles to the triangle V N P, and therefore as M G E C is parallel to the baſe, the cone is cut by the plane

C V N P

BOOK I. V N P paſſing through the axis of the cone, and perpendicular to the baſe N O P. Again as G L is at right angles to each of the two diameters M E, A D, the rectangle under M L, L E is equal to the rectangle under D L, L A, each of theſe rectangles (35. iii.) being equal to the ſquare of G L; and therefore (16. vi.) D L : L E : : M L : L A, and (6. vi.) the angle L D E is equal to the angle A M L, or (29. i.) V N P. The circle A F D E is therefore a ſubcontrary ſection.

Cor. A conic ſection neither parallel to the baſe of the cone, nor a ſubcontrary ſection, is not a circle.

DEFINITIONS.

XIV.

Fig. 10. The cones V A B D, V F G H, having the common vertex V, and whoſe ſuperficies are oppoſite, being generated by the ſame line as in the firſt Definition, are called *Oppoſite Cones*.

Cor. It is evident from this, and the ſecond Propoſition, that if either of the oppoſite cones be cut by a plane parallel to the baſe of either, the ſection will be a circle.

XV.

Fig. 15. If the plane V B E touch the conical ſuperficies in the ſide V B, and the cone V A B F be cut by the plane F D C parallel to the plane V B E, the ſection F D C, formed by the cutting plane and the cone, is called a *Parabola*.

XVI.

The plane V B E is called the *Vertical Plane to the Parabola*.

Cor. 1. As the cone may be indefinitely extended, it is evident that the parabola may alſo be indefinitely extended; and as the parabola does not ſurround the cone, it is evident that its curve does not include a ſpace.

Cor.

Cor. 2. An indefinite number of ſtraight lines parallel to V B may be drawn in the plane of the parabola. For the common ſection of any plane paſſing through V B, and any point in the parabola, with the parabola (16. xi.) will be parallel to V B. BOOK I.

XVII.

If the cone V A B C be cut by a plane, and if the ſection D K L H, formed by the plane and the cone, ſurround the cone, and is not a circle, it is called an *Ellipſe*. Fig. 16.

XVIII.

If the oppoſite cones V A B E, V M N, be cut by a plane V B E paſſing through the vertex V, and if they be alſo cut by a plane parallel to V B E, forming with the oppoſite cones the ſections F D C, Q R S; each of the ſections F D C, Q R S is called an *Hyperbola*, and when mentioned together they are called *Oppoſite Hyperbolas*. Fig. 17.

XIX.

The plane V B E is called the *Vertical Plane* to the Hyperbola, or Oppoſite Hyperbolas.

Cor. 1. It is evident, as each of the oppoſite cones may be indefinitely extended, that an hyperbola may be indefinitely extended; and that its curve does not include a ſpace.

Cor. 2. An indefinite number of ſtraight lines parallel to V B or V E may be drawn in the plane of the oppoſite hyperbolas. For the common ſection of any plane paſſing through V B, or V E, and any point in either hyperbola, with the plane of the hyperbolas, will be parallel (16. xi.) to V B or V E.

XX.

A ſtraight line in the plane of a conic ſection, which meets the curve, and which being produced both ways falls without it, is called a *Tangent*; but a ſtraight line

BOOK I. which meets the curve of a conic ſection in two points, or each of the oppoſite hyperbolas in one, is called *a Secant.*

SCHOLIUM.

Although only the Parabola, Ellipſe, and Hyperbola, are denominated Conic Sections, the attentive reader will readily perceive from the foregoing Propoſitions and Definitions, that five different Sections may be formed by the interſection of a cone and a plane varying its poſition. For if a ſtraight line parallel to the baſe be within the cone and remain fixed, and a plane move about it as an axis, when the plane paſſes through the vertex, the interſection of the cone and plane will be a triangle, as in the firſt Propoſition. When the plane has moved from the vertex, but ſtill cuts both the oppoſite cones, the ſection formed in each will be an hyperbola, as in the eighteenth Definition. When the plane, proceeding in its motion round the fixed ſtraight line, has arrived at a poſition parallel to that of a plane touching the cone in one of its ſides, the ſection which it then forms with the cone is a parabola, as in the fifteenth Definition. In any other poſition of the moving plane, beſides thoſe already mentioned, an ellipſe or circle will be formed with the cone, according to the circumſtances ſtated in the ſeventeenth Definition, and in the ſecond and fourth Propoſitions.

PROP. VI.

One ſtraight line, and one only, can be drawn to touch a conic ſection in a given point in the curve.

Fig. 14. Let G D H be a conic ſection, and let D be a given point in the curve; through D one ſtraight line, and only one, can be drawn to touch the ſection.

Let

Let V be the vertex of the cone V A C B, and through D draw V C a ſide of the cone, meeting the baſe in the point C. Draw C F (17. iii.) touching the baſe, and through V C, C F let a plane paſs, and let its line of common ſection with the plane of the ſection G D H be D E. Then the ſtraight line D E touches the ſection G D H, and no other ſtraight line can touch it in D.

For, as C F meets the baſe in the point C only, it is evident from the firſt Definition, that every ſtraight line, excepting V C, drawn from V to the tangent C F will fall without the ſuperficies V A C B. The plane paſſing through V C, C F, therefore, can only meet the ſuperficies in the ſtraight line V C, and the curve of the ſection G D H in the point D only. Conſequently as D E is in the plane paſſing through V C, C F, D E touches the ſection G D H, according to the twentieth Definition.

But no other ſtraight line can touch the ſection G D H in the point D. For, if it be poſſible, let D I touch the ſection, and then as D I meets the curve G D H only in the point D, it can meet the ſuperficies in that point only, and it will therefore touch the ſuperficies in D. Moreover as no ſtraight line, excepting D E the line of common ſection, can be in the plane of the ſection G D H and alſo in the plane V C F, and as D I, according to hypotheſis and the twentieth Definition, is in the plane of the ſection G D H, D I is not in the plane V C F. Let a plane be paſſed through the ſtraight lines V D C, D I, and let it cut the plane of the baſe in the ſtraight line K C. Then as the ſtraight line D I touches the ſuperficies, every point in it, excepting D, falls without the ſuperficies, according to the tenth Definition. It is therefore evident, from the firſt Definition, that every ſtraight line drawn from V, excepting V D C, in the plane paſſing through V D C, D I, will

BOOK I. fall without the ſuperficies. The plane paſſing through V D C, D I will therefore meet the ſuperficies in the ſtraight line V D C only; and conſequently K C will touch the baſe A C B. Again, as the planes V C K, V C F, cut one another in the ſtraight line V C, the ſtraight lines K C, C F are not in the ſame ſtraight line. The two ſtraight lines K C, C F therefore touch the circle A C B in the point C, which (16. iii.) is abſurd. Conſequently no other ſtraight line, beſides D E, can be drawn to touch the ſection G D H in the point D.

Cor. 1. If a ſtraight line as C F touch the baſe of the cone in the point C, and from V, the vertex, the ſide V C be drawn; a plane paſſing through C F, C V will touch the conical ſuperficies in the ſide V C; and it is evident from the firſt Definition (and 1. xi.) that this plane produced on the other ſide of V will touch the oppoſite ſuperficies in C V produced.

Cor. 2. If a ſtraight line as D E touch the conical ſuperficies, or a conic ſection G D H, and a ſide V D C of the cone be drawn through D the point of contact, a plane paſſing through this ſide of the cone and the tangent D E will touch the ſuperficies of the cone; and being produced beyond V, it will touch the oppoſite ſuperficies in V C produced. This is evident from the demonſtration of the Propoſition, and the preceding Cor.

Cor. 3. If the ſection G D H be an hyperbola, the tangent D E cannot meet the oppoſite hyperbola. For D E is the common interſection of the plane V C F and the plane of the ſection G D H, and, by the firſt Cor. the plane V C F touches the oppoſite ſuperficies in C V produced. It is therefore evident from the eighteenth Definition that the tangent D E cannot meet the oppoſite hyperbola.

PROP.

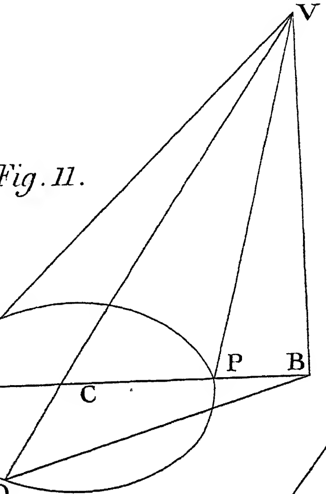


Fig. 11.

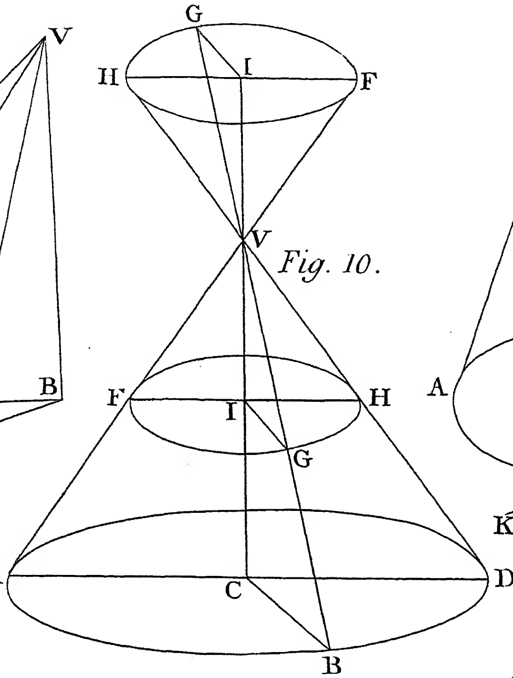


Fig. 10.

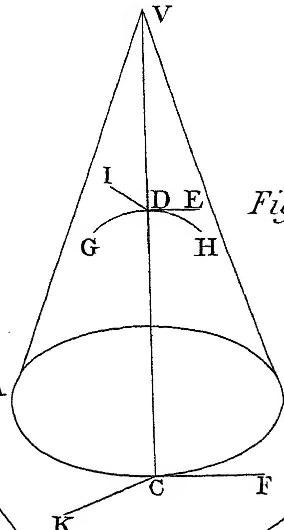
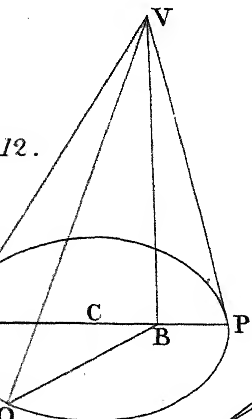


Fig. 14.



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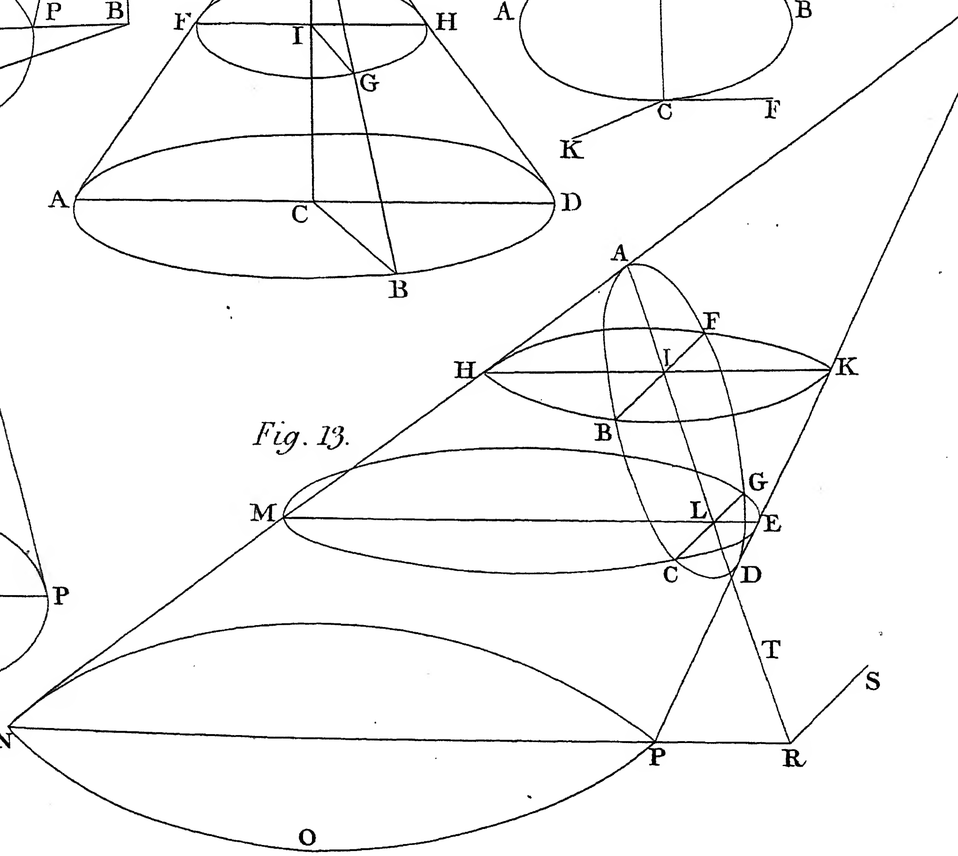

Fig. 13.

B4

PROP. VII.

BOOK I.

If a ſtraight line paſs through the vertex and fall without the oppoſite cones, two planes, and only two, can be drawn through it to touch the conical ſuperficies; and theſe planes will be on the oppoſite ſides of a plane paſſing through the ſtraight line, and cutting the baſe.

Let the ſtraight line V G paſs through V the vertex, and fall without the oppoſite cones V A M B, D V E; two planes, and only two, can be drawn through it to touch the ſuperficies; and if a plane paſs through V G, and cut the baſe in the ſtraight line C F, the tangent planes will be on the oppoſite ſides of the plane paſſing through V G, C F. Fig. 18. and 19.

Firſt let the ſtraight lines V G, C F be parallel. Let C F be biſected in L, and draw I L M at right angles to C F, and let it meet the circumference of the circle in the points M, I. Let a plane paſs through the ſtraight line V G and the point I, and this plane will touch the ſuperficies. For let it cut the plane of the baſe in the ſtraight line I K. Then as C F, V G are parallel, and as the plane of the baſe paſſes through C F, and the plane V I K paſſes through V G, the interſection I K of theſe planes will be parallel to C F, by the ſecond Lemma. But as I M biſects C F at right angles, it paſſes through the center of the circle (Cor. 1. iii.), and as I K, C F are parallel, and as the angle I L F is a right one, the angle K I L is alſo a right one (29. i.), and therefore I K (16. iii.) touches the circle in I. Conſequently, by Cor. 1. Prop. VI. the plane paſſing through V I, I K, or, as above, through V G, V I, touches the ſuperficies; and in the ſame manner it may be proved that the plane paſſing through V G and the point M touches the ſuperficies. Fig. 18.

Secondly, let the ſtraight line V G be not parallel to Fig. 19.

BOOK I.

C F, but let it meet it in K. Draw K I, K M (17. iii.) touching the bafe in I, M; and then the plane paffing through V G, I K, and alfo that paffing through V G and M K, will touch the fuperficies, by Cor. 1. Prop. VI.

It is evident, in either cafe, that no other plane, befides the above-mentioned two, can pafs through V G, and touch the fuperficies; and that one of thefe tangent planes is on the one fide, and the other on the oppofite fide of the plane paffing through V G, C F.

PROP. VIII.

If a conic fection furround the cone, two ftraight lines, and only two, parallel to one another, can be drawn to touch the fection; if the fection does not furround the cone, no ftraight line, parallel to a tangent, can be drawn to touch the fection; but if the fection be an hyperbola, one ftraight line, and one only, parallel to a tangent, can be drawn to touch the oppofite hyperbola.

Fig. 16.

Part I. Let the fection D H L K furround the cone V A F B, and let the ftraight line G D touch the fection in the point D: another ftraight line, and only one, parallel to G D, can be drawn to touch the fection.

For let V D A be the fide of the cone paffing through D, the point of contact, and let a plane pafs through V A, D G, and this plane will touch the conical fuperficies in the fide V A, by Cor. 2. Prop. vi. In this plane, and through V, the vertex of the cone, draw V T parallel to D G. Then V T will fall without the oppofite cones; and by Prop. VII. another plane can be paffed through V T touching the conical fuperficies. Let this plane touch the fuperficies in the fide V L B, and let its interfection with the plane of the fection D H L K be L I. Then as L I is in the plane touching the

the cone in the ſide V L B, it meets the conical ſuperficies in the point L only. It will therefore meet the curve of the ſection D H L K in the point L only, and conſequently it will touch the ſection: and as the plane T V L I paſſes through V T, and the plane of the ſection paſſes through D G parallel to V T, by the ſecond Lemma L I is parallel to D G. And as no other plane paſſing through V T can touch the conical ſuperficies, beſides the two T V A, T V B, it is evident, from the ſecond Lemma, that no other ſtraight line beſides L I, parallel to D G, can be drawn to touch the ſection D H L K.

Part II. Let F D C be a ſection which does not ſurround the cone, and let D G touch the ſection in the point D. No other ſtraight line, parallel to D G, can be drawn to touch the ſection. Fig. 15. and 17.

For let V B E be the vertical plane to the parabola, or hyperbola, as in the fifteenth, ſixteenth, eighteenth, and nineteenth Definitions. Let V D A be the ſide of the cone paſſing through D, the point of contact; and through V D A, D G let a plane paſs, and let it cut the vertical plane in the ſtraight line V T. Then, by Cor. 2. Prop. VI. the plane V D G will touch the conical ſuperficies, and (16. xi.) D G, V T will be parallel; and as V T is in the plane, touching the conical ſuperficies in the ſide V D A, it will fall without the oppoſite cones. Another plane, therefore, and only one, can be paſſed through V T to touch the conical ſuperficies, by Prop. VII. But when the ſection is a parabola, the other plane paſſing through V T and touching the ſuperficies is the vertical plane V B E, which is parallel to the parabola. When the ſection is an hyperbola, then the vertical plane V B E paſſes through V T, and cuts the baſe of the cone in the ſtraight line B E; and ſuppoſing T V L to be the other plane paſſing through V T, and

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and touching the conical ſuperficies, the planes T V L, V D G are on oppoſite ſides of V B E, by the ſeventh Propoſition. Conſequently the plane T V L cannot meet the hyperbola F D C. It therefore follows from the above, and the ſecond Lemma, that if the ſection does not ſurround the cone, no ſtraight line, parallel to a tangent, can be drawn to touch the ſection.

Fig. 17. Part III. Let F D C, Q R S be oppoſite hyperbolas, and let D G touch the hyperbola F D C in the point D. Then one ſtraight line, and only one, parallel to D G can be drawn to touch the oppoſite hyperbola Q R S.

For, every thing remaining as in the preceding part, through the ſides V A, V L, in which the planes T V A, T V L, paſſing through T V parallel to D G, touch the ſuperficies, let a plane be paſſed, cutting the vertical plane in the ſtraight line V W and the plane of the hyperbolas in the ſtraight line D H. Then as V W, H D, L V are in the ſame plane, and as (16. xi.) V W, H D are parallel, and L V meets V W, it will alſo meet H D, by the third Lemma. Let them meet in the point R. Then as the plane T V L touches the oppoſite cone M V N in L V produced, it will meet the plane of the hyperbolas in the point R. Let the interſection of theſe two planes therefore be R X; and as the plane of the hyperbolas paſſes through D G, and the plane T V L paſſes through T V parallel to D G, by the ſecond Lemma R X is parallel to D G; and being in the plane touching the conical ſuperficies, it will touch the hyperbola Q R S in the point R. It is alſo evident, for the ſame reaſons as are mentioned above, that no other ſtraight line parallel to R X, or D G, can be drawn to touch the hyperbola Q R S.

Cor. 1. If a ſtraight line touch a conic ſection, a ſtraight line drawn through any point within the ſection, and parallel to it, will meet the curve in two points.

points. For let every thing remain as in the demonſtration of the Propoſition, and let P be any point within the ſection. Through V T and the point P let a plane be paſſed, and let this plane cut the plane of the ſection in the ſtraight line H K; and, by the ſecond Lemma, H K will be parallel to G D the tangent, and alſo to V T. Now as the point P is within the ſection it is alſo within the cone, and therefore, by the firſt Propoſition, the plane paſſing through V T and the point P will cut the cone in two ſides; and as theſe two ſides and V T, H K are in the ſame plane, and V T, H K are parallel, H K will meet each of theſe two ſides, by the third Lemma; and as H K is in the plane of the ſection, it muſt meet the curve of the ſection in the ſame points in which it meets theſe two ſides of the cone. It is alſo evident from the Propoſition, and the preceding part of this Corollary, that a ſtraight line drawn through any point within the oppoſite hyperbola Q R S, and parallel to G D, will meet the curve Q R S in two points. (Fig. 15, 16, and 17.)

Cor. 2. If a ſtraight line meet the curve of a conic ſection in two points, two ſtraight lines may be drawn parallel to it to touch the ſection, if it ſurround the cone; but if the ſection does not ſurround the cone, only one ſtraight line parallel to a ſecant can be drawn to touch the ſection; and, if the ſection be an hyperbola, only one ſtraight line parallel to a ſecant can be drawn to touch the oppoſite hyperbola. For, let H, K be the points in which the ſecant H K meets the curve of the ſection; and through H K and V, the vertex of the cone, let a plane be paſſed, and, if the ſection ſurround the cone, draw V T in this plane parallel to H K. (Fig. 16.) Then as this plane, by the Cor. to Prop. I. can only cut the oppoſite ſuperficies in ſtraight lines drawn through V the vertex and the points H, K, it is evident,

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dent that V T muſt fall without the oppoſite cones. Conſequently, by Prop. VII. two planes can be paſſed through V T to touch the conical ſuperficies, one on each ſide of H K; and the interſections of theſe planes with the plane of the ſection will touch the ſection, and, by the ſecond Lemma, theſe tangents will be parallel to H K. If the ſection does not ſurround the cone, let the plane paſſing through the ſecant H K and V cut the vertical plane V B E in the ſtraight line V T, and, for the ſame reaſons as are mentioned above, V T will fall without the oppoſite cones. Then through V T two planes may be paſſed, by Prop. VII. touching the conical ſuperficies. But, according to the demonſtration of the Propoſition, only one of theſe planes can meet the parabola, and one of them can meet the hyperbola, and the other the oppoſite hyperbola; and, by the ſecond Lemma, G D, the interſection of the plane T V A with the plane of the ſection F D C, will be parallel to H K, and R X the interſection of the plane T V L with the plane of the oppoſite hyperbolas, will alſo be parallel to H K, and each of the ſtraight lines G D, R X muſt touch the ſection which it meets.

Fig. 15. and 17.

PROP. IX.

If a ſtraight line meet the curve of a conic ſection in two points, any ſtraight line parallel to it, drawn through a point within the ſame ſection, or, if the ſection be an hyperbola, within the oppoſite hyperbola, will alſo meet the curve of the ſection, in which it is drawn, in two points. And if a ſtraight line meet each of the curves of two oppoſite hyperbolas in one point, a ſtraight line parallel to it, drawn through any point in the plane of theſe ſections, will alſo meet each of the curves of theſe oppoſite hyperbolas in one point.

Part

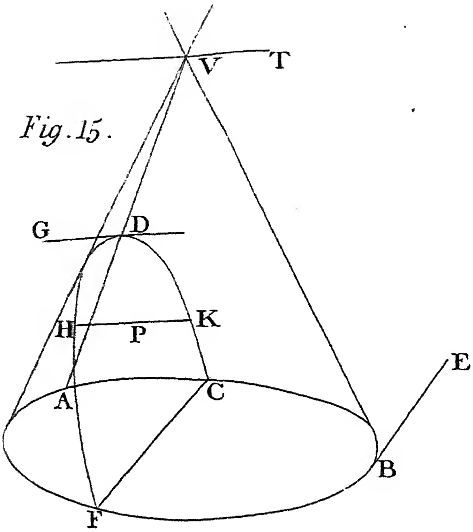


Fig. 15.

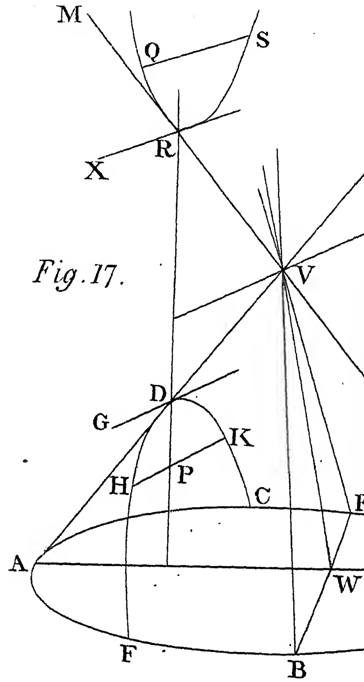


Fig. 17.

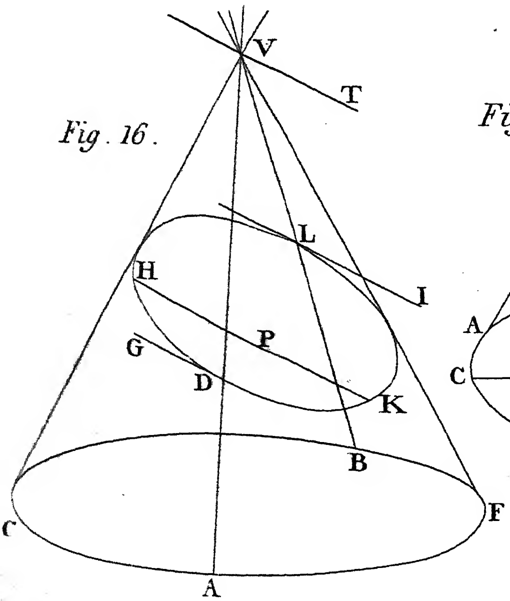


Fig. 16.

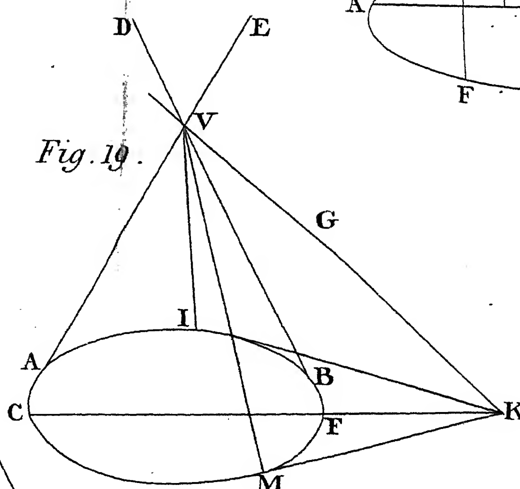
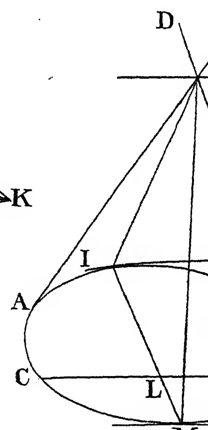


Fig. 19.



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B

Part I. Let A C D B be a conic ſection, and let the ſtraight line C D meet the curve in the points C, D; a ſtraight line, as A B, drawn parallel to C D through any point P within the ſection will meet the curve in two points. Fig. 20.

For let V be the vertex of the cone V E G H F, in which the ſection is formed, and through C D and V let a plane be paſſed, and let it cut a plane paſſing through A B and V in the ſtraight line V T. Then the plane paſſing through C D and V muſt cut the cone in the ſides V C, V D, and, by the ſecond Lemma, V T is parallel to C D and alſo to A B. Again, as V T is in the plane V C D, and as this plane, by Cor. Prop. I. cuts the oppoſite ſuperficies only in V C, V D, or in theſe lines produced, it follows that V T falls without the oppoſite cones. The plane paſſing through V T, A B will therefore cut the cone in two ſides, and, by the third Lemma, A B will meet each of theſe two ſides in the ſuperficies of the cone, and being in the plane of the ſection, it muſt meet the curve in the ſame points. If the ſection be an hyperbola, it is evident, for the ſame reaſons, that a ſtraight line drawn parallel to C D, through any point within the oppoſite hyperbola will meet the curve in two points.

Part II. Let A C, B D be two oppoſite hyperbolas, and let the ſtraight line C D meet the curve of one of them in the point C, and the curve of the other in the point D. Through the point P in the plane of theſe ſections let the ſtraight line A B be drawn parallel to C D; A B will meet each of the oppoſite hyperbolas A C, B D in one point. Fig. 21.

For let V be the vertex of the oppoſite cones, in which the hyperbolas are formed, and through C D and V let a plane be paſſed, and let it cut a plane paſſing through A B and V in the ſtraight line V T. Then the

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the plane paſſing through C D and V muſt cut the oppoſite cones in the ſides V C, V D; and, by the ſecond Lemma, V T is parallel to C D and alſo to A B. Again, as V T is in the plane V C D, and as this plane, by Cor. Prop. I. cuts the oppoſite ſuperficies only in V C, V D, or in theſe lines produced, it follows that V T falls within the oppoſite cones. The plane paſſing through V T, A B will therefore cut the oppoſite cones in two ſides; and from the above, and the third Lemma, A B will meet one of theſe two ſides in the one ſuperficies, and the other in the oppoſite ſuperficies. But as A B is in the plane of the oppoſite hyperbolas A C, B D, it muſt meet the curve of one of the hyperbolas and the ſuperficies, in which this hyperbola is, in the ſame point. The ſtraight line A B will therefore meet the curve of each of the oppoſite hyperbolas in one point.

Cor. If a ſtraight line as C D meet the curve of a conic ſection in two points, it will fall wholly within the ſection, but being produced it will fall without the ſection. If a ſtraight line meet each of the curves of oppoſite hyperbolas in one point, it will fall wholly without the hyperbolas, but being produced it will fall on the one ſide within one hyperbola, and on the other within the oppoſite hyperbola. For it is evident from the demonſtration of the Propoſition that C D in Fig. 20. is within the cone, and that being produced it muſt fall without it; and in Fig. 21. it is evident that C D is without the oppoſite cones, and that being produced it falls on the one ſide within one cone, and on the other ſide within the oppoſite cone.

PROP. X.

If a ſtraight line cut either or both of the oppoſite conical ſuperficies, and meet a ſtraight line which is parallel to the baſe of the cone, and which cuts either of the oppoſite ſuper-

superficies, the rectangle under the segments of the first mentioned line will be to the rectangle under the segments of the other in the same ratio, wherever the point of concourse may be in the first mentioned line *. BOOK I.

Let the straight line F G H cut either or both of the opposite superficies in the points G, H, and meet, in the point F, the straight line F D E parallel to the base of the cone, and cutting either of the opposite superficies in the points E, D; the rectangle under G F, F H will be to the rectangle under D F, F E in the same ratio, wherever the point of concourse F may be in the straight line F G H. Fig. 26, 27, 28.

Case 1. If the straight line F G H be also parallel to the plane of the base, then the section G D H E, formed by the cone and the plane passing through F G H, F E D will be a circle, by the fourth Lemma, and Prop. II. and therefore (35 or Cor. 36. iii.) the rectangle under G F, F H will be to the rectangle under D F, F E in the ratio of equality. Fig. 26.

Case 2. Let F G H be not parallel to the base of the cone. Through F E D let a plane be passed parallel to the base, and let the section formed by it with the cone be the circle D E I K, as in the second Proposition. Through the points G, H, and V, the vertex of the cone, let a plane be passed, and let it cut the superficies in the sides A V I, B V K, the plane of the base in the straight line A B L, and the plane of the circle D E I K in the straight line I F K; and in the plane V G H draw V L parallel to G H, and let it meet A B L in the point L. Fig. 27, 28.

* When two secants meet one another, the segments of either of the two are its parts between the point of concourse and the points in which it meets the superficies; and if a tangent meet a secant, or another tangent, its magnitude is limited by the point of concourse, and its point of contact.

Then

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Then as the ſtraight lines A B L, I F K (16. xi.) are parallel to one another, and the ſtraight lines V L, F G H alſo parallel to one another, in the triangles V L B, H F K, the angles (29. i.) B V L, V B L in the one are equal to the angles K H F, H K F in the other, each to each; and in the triangles V L A, G F I, the angles V A L, A V L in the one are equal to the angles G I F, I G F in the other, each to each. The triangles V L B, H F K are therefore equiangular to one another, as are alſo the triangles V L A, G F I to one another. Hence (4. vi.)

V L : L A : : G F : F I, and

V L : L B : : H F : F K, and therefore, by the fifth Lemma, $VL^2 : AL \times LB :: GF \times FH : IF \times FK$. But (35 or Cor. 36. iii.) $IF \times FK = DF \times FE$, and conſequently,

$$VL^2 : AL \times LB :: GF \times FH : DF \times FE.$$

Cor. 1. If the ſtraight line F G H meet the ſtraight line F P parallel to the baſe, and touching either ſuperficies in the point P, the rectangle under G F, F H will be to the ſquare of F P in the ſame ratio, wherever the point of concourſe may be in the line F G H, as is evident from the above (and 36. iii.) And if the ſtraight line M V, paſſing through V the vertex, be parallel to the baſe, and meet G H in M, M V is to be conſidered as a tangent; for as above, by ſimilar triangles,

V L : L A : : G M : M V, and

V L : L B : : H M : M V, and therefore, by the fifth Lemma, $VL^2 : AL \times LB :: GM \times MH : MV^2$.

Cor. 2. If a ſtraight line cut either or both the oppoſite ſuperficies, and meet a ſtraight line parallel to the baſe, and touching or cutting either ſuperficies; the rectangle under the ſegments of the firſt mentioned line will be to the ſquare of the tangent, or the rectangle under the ſegments of the ſecant which it meets

in

in a conftant ratio, wherever the point of concourfe may be in the firft mentioned line. For this ratio will be either that of equality, as in the firft cafe of the demonftration, or it will be that of VL^2 to $AL \times LB$, as in the fecond cafe, and in the preceding Cor. BOOK I.

PROP. XI.

If a ftraight line touch either of the oppofite fuperficies, and meet a ftraight line parallel to the bafe of the cone, and which cuts either of the oppofite fuperficies, the fquare of the tangent will be to the rectangle under the fegments of the fecant in the fame ratio, wherever the point of concourfe may be in the tangent.

Let the ftraight line TF touch either of the oppofite fuperficies in the point T, and meet in the point F the ftraight line FE, parallel to the bafe of the cone, and cutting either of the oppofite fuperficies in the points G, E; the fquare of the tangent TF will be to the rectangle under GF, FE, in the fame ratio, wherever the point F may be in the tangent TF. Fig. 22. 23.

Cafe 1. If the tangent TF be alfo parallel to the plane of the bafe, then the fection TGE, formed by the cone and the plane paffing through TF, FE will be a circle, by the fourth Lemma, and Prop. II, and the fquare of TF (36. iii.) will be to the rectangle GF, FE in the ratio of equality. Fig. 22.

Cafe 2. Let TF be not parallel to the bafe of the cone. Through FE let a plane be paffed parallel to the bafe, and let the fection formed by it with the cone be the circle DGE, as in the fecond Propofition. Through the tangent TF and V, the vertex of the cone, let a plane be paffed, and let it touch the fuperficies in the fide VTD, according to Cor. 2. Prop. VI. and let it cut the plane of the bafe in the ftraight line CL, and Fig. 23.

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the plane of the circle D G E in the ſtraight line D F. In the plane V C L draw V L parallel to T F, and let it meet the baſe in the point L. Then as the ſtraight lines C L, D F (16. xi.) are parallel to one another, and the ſtraight lines V L, T F alſo parallel to one another, in the triangles V C L, T D F, the angles (29. i.) C V L, V C L in the one are equal to the angles D T F, T D F in the other, each to each.

The triangles V C L, T D F are therefore equiangular, and conſequently (4. vi.)

$VL : LC :: TF : FD$, and by the fifth Lemma,

$VL^2 : LC^2 :: TF^2 : FD^2$.

But (36. iii.) FD^2 is equal to the rectangle, under G F, F E, and conſequently $VL^2 : LC^2 :: TF^2 : GF \times FE$.

Cor. 1. If a ſtraight line, as T F, touch either of the oppoſite ſuperficies in T, and meet a ſtraight line, as F D, parallel to the baſe of the cone, and which touches either ſuperficies as in D; the ſquare of T F will be to the ſquare of F D in the ſame ratio, wherever the point of concourſe F may be in T F. If M V, paſſing through V the vertex, be parallel to the baſe and meet T F in M, then M V is to be conſidered as a tangent; and as above $VL^2 : LC^2 :: TM^2 : MV^2$.

Cor. 2. If a ſtraight line touch either of the oppoſite ſuperficies, and meet a ſtraight line parallel to the baſe, and touching or cutting either ſuperficies; the ſquare of the firſt mentioned tangent will be to the ſquare of the tangent, or the rectangle under the ſegments of the ſecant which it meets in a conſtant ratio, wherever the point of concourſe may be in the firſt mentioned tangent. For this ratio will be that of equality, as in the firſt caſe of the demonſtration, or it will be that of VL^2 to LC^2, as in the ſecond caſe.

PROP.

PROP. XII.

If the firſt of two ſtraight lines be parallel to the ſecond and touch or cut either or cut both of the oppoſite ſuperficies, and if the ſecond alſo touch or cut either, or cut both of the oppoſite ſuperficies, and if each of the two meet a ſtraight line parallel to the baſe of the cone and touching or cutting either ſuperficies; then the ſquare of the firſt, if a tangent, or the rectangle under its ſegments, if a ſecant, will be to the ſquare of the tangent or the rectangle under the ſegments of the ſecant which it meets, as the ſquare of the ſecond, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of the tangent, or the rectangle under the ſegments of the ſecant which it meets.

Caſe 1. If the firſt and ſecond ſtraight lines, parallel to one another, be alſo parallel to the plane of the baſe, then the ſquare of the firſt, if a tangent, or the rectangle under its ſegments, if a ſecant, will be to the ſquare of the tangent, or the rectangle under the ſegments of the ſecant which it meets in a ratio of equality, as in the firſt caſe of the demonſtration of Prop. X. and XI. And, for the ſame reaſons, the ſquare of the ſecond, if a tangent, or the rectangle under its ſegments, if a ſecant, will be to the ſquare of the tangent, or the rectangle under the ſegments of the ſecant which it meets in a ratio of equality. In this caſe therefore the Propoſition is evident.

Caſe 2. Let the firſt and ſecond ſtraight lines, parallel to one another, be not parallel to the plane of the baſe. Suppoſe a plane to paſs through each of the two parallel lines, and V the vertex, and then theſe planes will cut one another, and their line of common ſection will paſs through V, and, by the ſecond Lemma, it will be parallel to each of the two ſtraight lines parallel to

 one

BOOK I. one another. Let V L be their line of common ſection, as in Fig. 27. 28. and 23. and let it meet the baſe in the point L. Then the planes paſſing through the two parallel lines and V L, will cut the plane of the baſe in ſtraight lines paſſing through L; and each of theſe lines of common ſection with the plane of the baſe will cut or touch the baſe, as in the ſecond caſe of the demonſtration of the tenth and eleventh Propoſitions; and the rectangle under the ſegments of either, if a ſecant, or its ſquare, if a tangent, will be (35 and 36. iii.) equal to the rectangle under A L, L B, L A being a ſtraight line cutting the baſe in the points B, A. Conſequently by Cor. 2. Prop. X. and Cor. 2. Prop. XI. the ſquare of the firſt ſtraight line, if a tangent, or the rectangle under its ſegments, if a ſecant, will be to the ſquare of the tangent, or the rectangle under the ſegments of the ſecant, which it meets as the ſquare of V L to the rectangle under A L, L B. For the ſame reaſon, the ſquare of the ſecond, if a tangent, or the rectangle under its ſegments, if a ſecant, will be to the ſquare of the tangent, or the rectangle under the ſegments of the ſecant, which it meets as the ſquare of V L to the rectangle under A L, L B. Hence (11. v.) if the firſt of two ſtraight lines be parallel, &c.

SCHOLIUM.

As every point in the curve of a Conic Section is alſo in the conical ſuperficies, it is evident that all the Propoſitions demonſtrated concerning ſtraight lines touching or cutting the conical ſuperficies, or oppoſite ſuperficies, may be transferred to ſtraight lines, which in the ſame manner touch or cut a conic ſection, or oppoſite hyperbolas.

PROP.

PROP. XIII.

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Fig. 24.

If there be four ſtraight lines in the plane of a conic ſection, and if A B *the firſt meet* C B *the ſecond, and* D E *the third meet* F E *the fourth, and if the firſt be parallel to the third, and the ſecond to the fourth, and if each of them either touch or cut a conic ſection, or cut oppoſite hyperbolas; then the ſquare of* A B, *if a tangent, or the rectangle under its ſegments, if a ſecant, will be to the ſquare of* C B, *if a tangent, or the rectangle under its ſegments, if a ſecant, as the ſquare of* D E, *if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of* F E, *if a tangent, or the rectangle under its ſegments, if a ſecant.*

Caſe 1. If the ſtraight lines A B, C B, and conſequently D E, F E, be each parallel to the baſe of the cone in which the ſection was formed, or the baſe of the oppoſite cone, the ſection muſt be a circle, as in the firſt caſe of Prop. X. or Prop. XI, and the ratio above ſtated will be that of equality.

Caſe 2. Let A B, D E be not parallel to the baſe of the cone, in which, or in which and its oppoſite, the ſection or oppoſite hyperbolas were formed; but let C B, F E be parallel to the baſe, and then the Propoſition is evident from the twelfth Propoſition.

Caſe 3. Let neither A B nor C B, and conſequently neither D E nor F E, be parallel to the baſe of the cone; but ſuppoſe B G, E H to be ſtraight lines parallel to the baſe of the cone in which the ſection, or in which and the oppoſite cone the oppoſite hyperbolas were formed; and let B G, E H touch or cut either of the oppoſite conical ſuperficies. Then by the twelfth Propoſition, the ſquare of A B, if a tangent, or the rectangle under its ſegments, if a ſecant, will be to the ſquare of B G, if a tangent, or the rectangle under its ſegments, if a ſe-

 cant,

BOOK I.

cant, as the ſquare of D E, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of E H, if a tangent, or the rectangle under its ſegments, if a ſecant. Again, by the twelfth Propoſition and inverſion, the ſquare of B G, if a tangent, or the rectangle under its ſegments, if a ſecant, is to the ſquare of C B, if a tangent, or the rectangle under its ſegments, if a ſecant, as the ſquare of E H, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of F E, if a tangent, or the rectangle under its ſegments, if a ſecant. Conſequently,

$$\left.\begin{matrix}t.\,AB^{2}\\ \text{or}\\ \text{ſ.}\,AB^{r}\end{matrix}\right\} : \left\{\begin{matrix}t.\,BG^{2}\\ \text{or}\\ \text{ſ.}\,BG^{r}\end{matrix}\right\} : \left\{\begin{matrix}t.\,CB^{2}\\ \text{or}\\ \text{ſ.}\,CB^{r}\end{matrix}\right.$$

$$\left.\begin{matrix}t.\,DE^{2}\\ \text{or}\\ \text{ſ.}\,DE^{r}\end{matrix}\right\} : \left\{\begin{matrix}t.\,EH^{2}\\ \text{or}\\ \text{ſ.}\,EH^{r}\end{matrix}\right\} : \left\{\begin{matrix}t.\,FE^{2}\\ \text{or}\\ \text{ſ.}\,FE^{r}\end{matrix}\right.$$

The ſquare of A B therefore (22. v.) if a tangent, or the rectangle under its ſegments, if a ſecant, is to the ſquare of C B, if a tangent, or the rectangle under its ſegments, if a ſecant, as the ſquare of D E, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of F E, if a tangent, or the rectangle under its ſegments, if a ſecant.

Cor. If A B, C B, D E, F E be tangents, then it is evident (22. vi.) that A B : C B : : D E : F E.

PROP. XIV.

Any ſtraight line parallel to the ſide of a cone, provided it be not in the plane touching the cone in that ſide, will meet one of the oppoſite ſuperficies in one point, and in one point only.

Fig. 29. 30. Let the ſtraight line D C be parallel to V B a ſide of the cone V A M B, but not ſituated in the plane touching

ing the cone in the ſide V B; the ſtraight line D C will meet one of the oppoſite ſuperficies in one point, and in one point only. BOOK I.

Let a plane paſs through the parallels D C, V B, and as by hypotheſis D C is not in the plane touching the cone in the ſide V B, the plane paſſing through D C, V B muſt cut the cone. Let it cut the oppoſite ſuperficies therefore in the ſtraight lines A V, B V. Then as A V meets B V in V the vertex, and as it is in the ſame plane with the parallels V B, D C, by the third Lemma A V, or A V produced, muſt alſo meet D C. Let them meet in D. Then D C muſt meet one of the ſuperficies in D, and as it is parallel to V B, it is evident it cannot meet the other ſuperficies. It is alſo evident, that it can meet one of the oppoſite ſuperficies in one point only; for on one ſide of D it is entirely within the ſuperficies, and on the other entirely without it.

PROP. XV.

If a ſtraight line parallel to a ſide of the cone cut either of the oppoſite conical ſuperficies, and meet two ſtraight lines parallel to the baſe of the cone, and which cut either ſuperficies; the ſegments of the firſt mentioned line, between the ſuperficies and the points of concourſe, will be to one another as the rectangles under the ſegments of the ſecants which it meets.

Let the ſtraight line D C, parallel to V B a ſide of the cone V A M B, cut either of the oppoſite ſuperficies in the point D, and meet in the points E, R the ſtraight lines E I F, L R T, which are parallel to the baſe of the cone, and cut either ſuperficies in the points I, F, and L, T; then the ſegment D E is to the ſegment D R as the rectangle under E I, E F to the rectangle under L R, R T. Fig. 29. 30.

 For

BOOK I.

For through the parallels D C, V B let a plane paſs, and let it cut the plane of the baſe in the ſtraight line A C B, and the ſuperficies in A V, B V. Through each of the ſtraight lines L R T, E I F let a plane paſs parallel to the baſe A M B, and let O L N T, I F H G be the circles formed, as in the ſecond Propoſition. Let E G H, O R N be the interſections of theſe circles and the plane paſſing through A V, B V. Let E G H meet A V in G and B V in H; and let O R N meet A V in O and B V in N. Then (16. xi.) the ſtraight lines E G H, O R N, A C B are parallel, and therefore (34. i.) E H, C B, R N are equal to one another; the angle D G E (29. i.) is equal to the angle D A C, and the angle D E G to the angle D C A. Hence (4. vi.) D E : E G : : D C : A C, and D R : O R : : D C : A C; and therefore (11. v.) D E : E G : : D R : O R. and (16. v.) D E : D R : : E G : O R. Conſequently (1. vi.) D E : D R : : E G × E H : O R × R N.

But (35. and 36. iii.) E G × E H = E I × E F, and O R × R N = L R × R T; and therefore D E : D R : : E I × E F : L R × R T.

Cor. If a ſtraight line parallel to a ſide of the cone cut either of the oppoſite conical ſuperficies, and meet two ſtraight lines parallel to the baſe, and which meet either ſuperficies; its ſegment between the ſuperficies, and the firſt of the two parallel to the baſe, will be to its ſegment between the ſuperficies and the ſecond, as the ſquare of the firſt, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of the ſecond, if a tangent, or the rectangle under its ſegments, if a ſecant. For every thing remaining as above, if E P be parallel to the baſe, and touch either ſuperficies in P, E P will be in the plane of the circle I F H G and (36. iii.) the ſquare of E P will be equal to the rectangle under E I, E F. And, for the ſame reaſons, if the point

point R were without the circle, the ſquare of a ſtraight line parallel to the baſe, drawn from R and touching either ſuperficies would be equal to the rectangle under L R, R T. If a ſtraight line V K parallel to the baſe paſs through V, the vertex, and meet C D in K, V K is to be conſidered as a tangent. For D K : K V : : D C : C A, and D R : O R : : D C : C A. Hence (11. v.) D K : K V : : D R : O R, and (16. v.) D K : D R : : K V : O R. But K V, R N being parallel to the baſe are parallel to one another, and therefore (34. i.) K V, R N are equal, and conſequently (1. vi.) D K : D R : : K V^2 : O R × R N. BOOK I.

PROP. XVI.

If a ſtraight line cutting a parabola or hyperbola be parallel to a ſide of the cone in which the ſection is formed, and if it meet two ſtraight lines which are parallel to one another, and meet the ſame ſection, or the oppoſite hyperbolas; its ſegment between the curve and the firſt of the two parallels will be to its ſegment between the curve and the ſecond, as the ſquare of the firſt, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of the ſecond, if a tangent, or the rectangle under its ſegments, if a ſecant.

Suppoſe the ſtraight line B C to cut the curve of a parabola or hyperbola in the point A, and to be parallel to a ſide of the cone in which the ſection is formed, and let it meet the ſtraight lines B D, C E which are parallel to one another, and meet the ſame ſection, or the oppoſite hyperbolas; then A B is to A C as the ſquare of B D, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of C E, if a tangent, or the rectangle under its ſegments, if a ſecant. Fig. 25.

For, as B C cuts the curve of the parabola or hyperbola

BOOK I.

bola in the point A, it will cut the conical ſuperficies in which the curve is formed in the ſame point; and the ſtraight lines B D, C E will meet the conical ſuperficies in the ſame points in which they meet the curve of the ſection, or the curves of the oppoſite hyperbolas. If therefore B D, C E be parallel to the baſe of the cone, the Propoſition is evident from the Cor. to Prop. XV. but if they are not parallel to the baſe of the cone, let B F, C G be parallel to the baſe of the cone, and let them meet the ſame or the oppoſite conical ſuperficies. Then by the Cor. to Prop. XV. A B is to A C as the ſquare of B F, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of C G, if a tangent, or the rectangle under its ſegments, if a ſecant. Again, by Prop. XII. (and 16. v.) the ſquare of B F, if a tangent, or the rectangle under its ſegments, if a ſecant, is to the ſquare of C G, if a tangent, or the rectangle under its ſegments, if a ſecant, as the ſquare of B D, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of C E, if a tangent, or the rectangle under its ſegments, if a ſecant. Conſequently (11. v.) A B is to A C as the ſquare of B D, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of C E, if a tangent, or the rectangle under its ſegments, if a ſecant.

PROP. XVII.

If two ſtraight lines meeting one another touch a conic ſection, or oppoſite hyperbolas, and if a ſecant parallel to one of them meet the other and the ſtraight line joining the points of contact, the rectangle under the ſegments of the ſecant between the curve and the tangent will be equal to the ſquare of its ſegment between the tangent and the line joining the points of contact.

Let

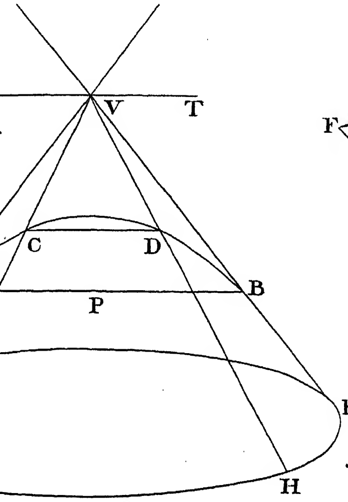


Fig. 23.

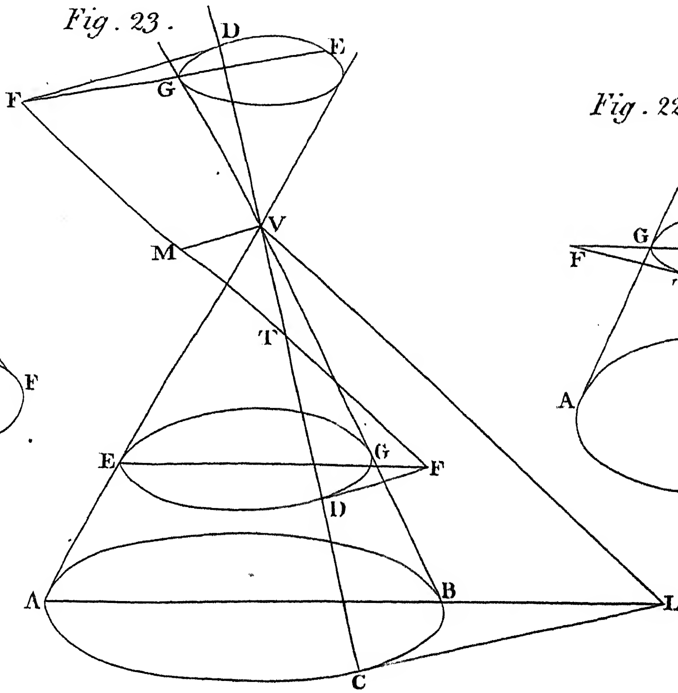
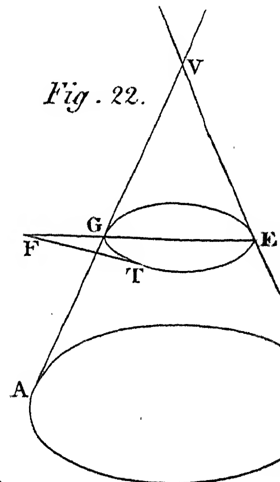


Fig. 22.



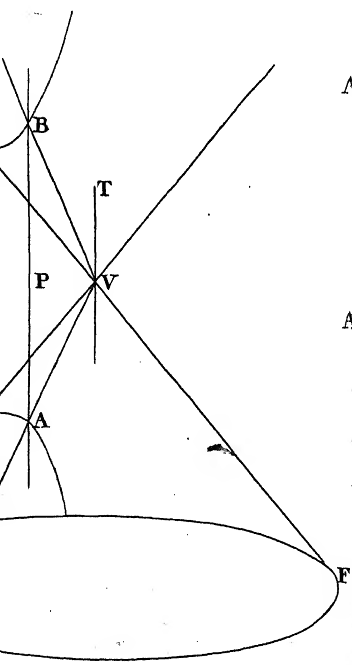


Fig. 24.

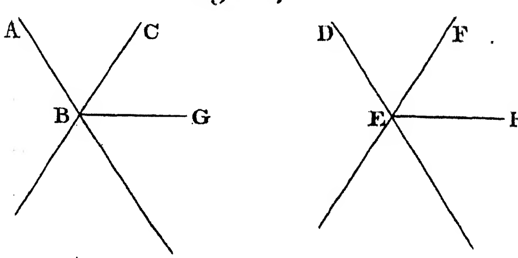
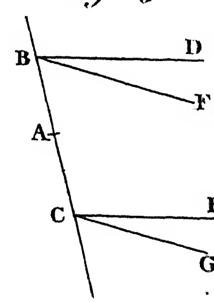


Fig. 25.



J

in the points A, C, and let the straight line D P, parallel to C F, cut the section or either of the opposite hyperbolas in D, P, and meet the tangent A F in G, and the straight line A C, joining the points of contact, in B; then the rectangle under D G, G P is equal to the square of G B. Fig. 31, 32, 33, 34.

For by Prop. XIII. $DG \times GP : AG^2 :: CF^2 : AF^2$. But as C F, B G are parallel, by Lemma V. (and 4. vi.) $CF^2 : AF^2 :: GB^2 : AG^2$, and therefore (11. v.) $DG \times GP : AG^2 :: GB^2 : AG^2$. Consequently (14. v.) $DG \times GP = GB^2$.

Cor. 1. The rest remaining as above, if from any point E in the tangent A F, there be drawn the straight line E H parallel to the tangent C F, and meeting A C in H, and if from the same point E there be drawn any straight line E I L, cutting the section or opposite hyperbolas in I and L; then the rectangle under I E, E L and the square of E H will be to one another as the squares of the tangents, or the rectangles under the segments of the secants meeting one another and parallel to I L, E H. For from the point G draw G N parallel to I L, and let it cut the curve or curves in M, and N. Then by Prop. XIII. $IE \times EL : MG \times GN :: AE^2 : AG^2$; and by similar triangles, and this Proposition, $AE^2 : AG^2 :: EH^2 : GB^2$ or its equal $DG \times GP$. Hence (11. v.) $IE \times EL : MG \times GN :: EH^2 : DG \times GP$, and, by alternation, $IE \times EL : EH^2 :: MG \times GN : DG \times GP$. Consequently, by Prop. XIII. (and 11. v.) the Cor. is evident.

Cor. 2. If a straight line B H cut a conic section, or opposite hyperbolas in D, G, and meet in the points B, H two straight lines A B, C H which touch the section or opposite hyperbolas in A, and C; and if B H meet Fig. 35. 36.

A C

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A C joining the points of contact in E; the rectangle under D B, B G will be to the rectangle under G H, H D as the square of B E to the square of H E. For if the tangents A B, C H be parallel, the triangles (29. i.) A B E, C H E will be equiangular, and therefore, by the fifth Lemma, (and 4. vi.) in this case $AB^2 : CH^2 :: BE^2 : HE^2$; and by Prop. XIII. $AB^2 : CH^2 :: DB \times BG : GH \times HD$. In this case therefore (11. v.) the Cor. is evident. But if the tangents be not parallel, through H draw the straight line L K parallel to A B, and let it meet A C in K, and the curve or curves in the points L, M. Then the triangles A B E, K H E are (29. i.) equiangular, and as above $AB^2 : HK^2 :: BE^2 : HE^2$. But by this Proposition HK^2 is equal to $LH \times HM$, and therefore $AB^2 : LH \times HM :: BE^2 : HE^2$; and by Prop. XIII. $AB^2 : LH \times HM :: DB \times BG : GH \times HD$. Consequently (11. v.) $DB \times BG : GH \times HD :: BE^2 : HE^2$.

A GLO-

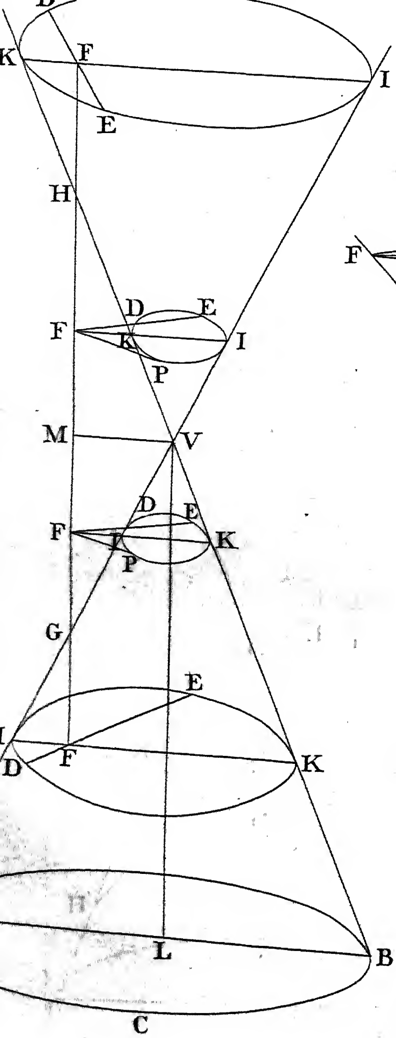
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Fig. 27.

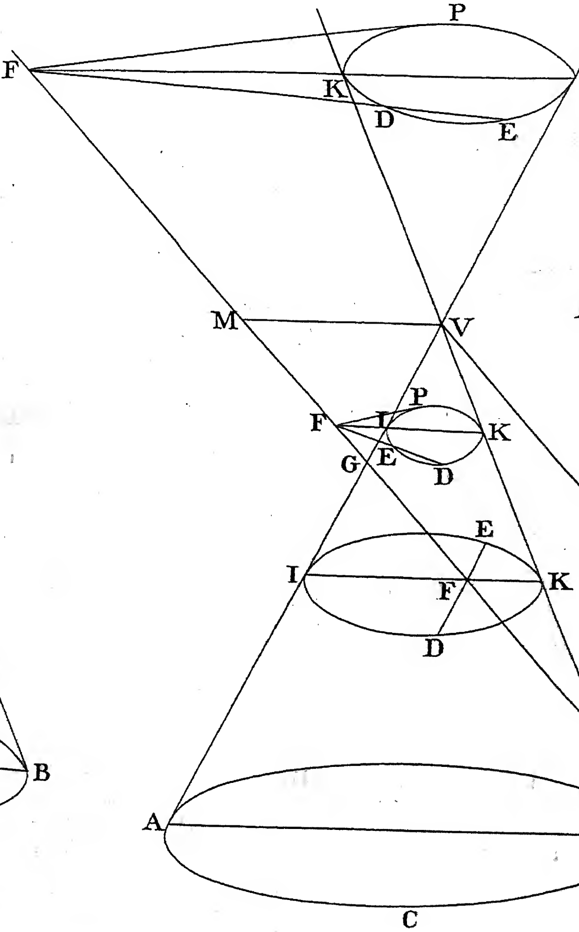
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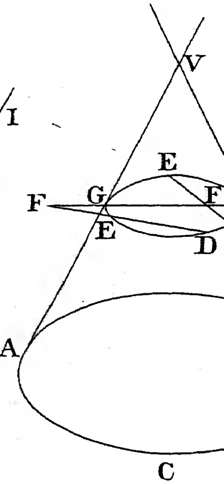
Fig. 2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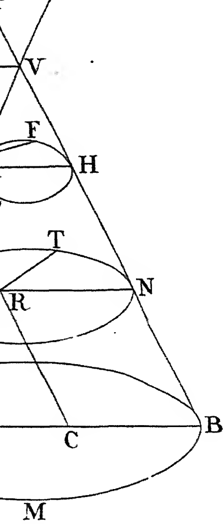


Fig. 31.

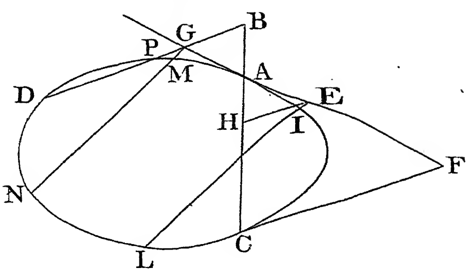


Fig. 30.

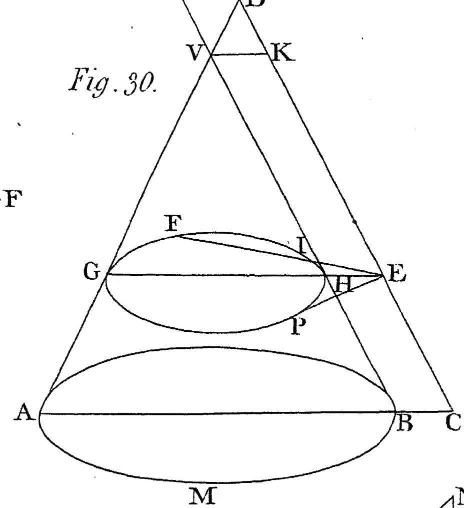


Fig. 32.

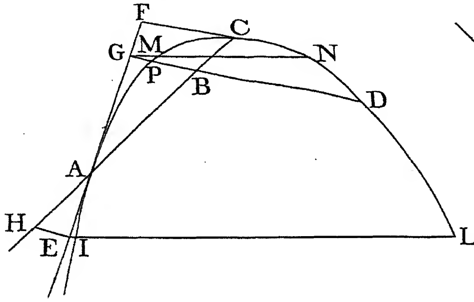


Fig. 33.

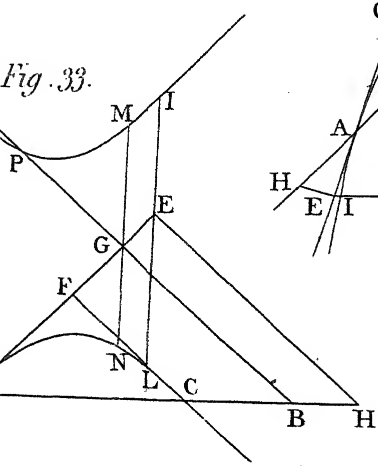
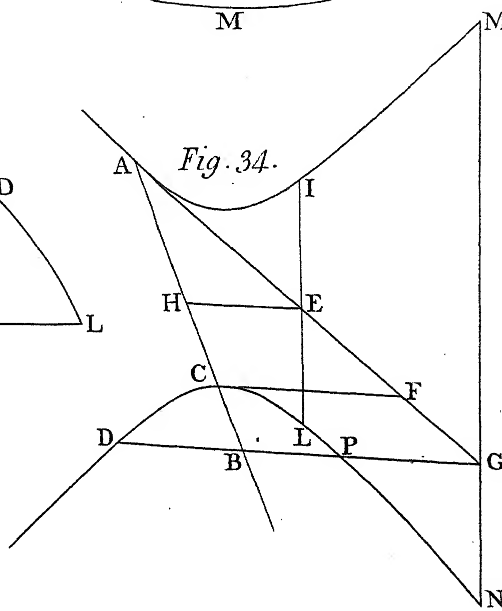


Fig. 34.



J. Basire s

A

GEOMETRICAL TREATISE

OF

CONIC SECTIONS.

BOOK II.

Of the Ellipse and Hyperbola.

DEFINITIONS.

I.

THAT point within an ellipſe or between oppoſite hyperbolas, in which every ſtraight line, paſſing through it and terminated by the curve or oppoſite curves, is biſected, is called *the Center* of the ellipſe, or *the Center* of the hyperbola or oppoſite hyperbolas.

II.

Any ſtraight line paſſing through the center of an ellipſe, and terminated by the curve, is called *a Diameter* of the ellipſe.

III.

A ſtraight line paſſing through the center of oppoſite hyperbolas, and terminated by the oppoſite curves, is called

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called *a Tranſverſe Diameter* of the oppoſite hyperbolas, or of either of the oppoſite hyperbolas. And a ſtraight line paſſing through the center of oppoſite hyperbolas, and biſecting a ſtraight line not paſſing through the center, and terminated by the oppoſite curves, is called *a Second Diameter* of the oppoſite hyperbolas, or of either of the oppoſite hyperbolas.

IV.

Any ſtraight line not paſſing through the center of an ellipſe, or oppoſite hyperbolas, terminated by the curve of the ellipſe or either hyperbola, or by the oppoſite curves, and biſected by a diameter, is called *a Double Ordinate* to the biſecting diameter; and its half is ſimply called *an Ordinate* to it.

V.

The points in which any diameter of an ellipſe meets the curve, or in which any tranſverſe diameter of oppoſite hyperbolas meets the oppoſite curves, are called *the Vertices* of the diameter; and the ſegments of a diameter, between an ordinate and its vertices, are called *Abſciſſes*.

VI.

Two diameters of an ellipſe, or oppoſite hyperbolas, of which each biſects all ſtraight lines terminated by the curve, or oppoſite curves, and parallel to the other, are called *Conjugate Diameters*.

VII.

A diameter of an ellipſe, or oppoſite hyperbolas, which cuts its ordinates at right angles is called *an Axis* of the ellipſe, hyperbola, or oppoſite hyperbolas.

PROP. I.

If two parallel ſtraight lines touch an ellipſe, or oppoſite hyperbolas, the ſtraight line joining the points of contact will

will bisect any straight line parallel to them, and terminated by the curve of the ellipse, or by the curve of either of the opposite hyperbolas. BOOK II.

Let the two parallel straight lines G H, L M touch the ellipse A D B E, or the opposite hyperbolas F B N, I A K, in the points B, A; the straight line A B, joining the points of contact, will bisect any straight line F N, parallel to the tangents, and terminated by the curve of the ellipse, or by the curve of either of the opposite hyperbolas. Fig. 37. 38.

Let F N meet the curve in the points F, N, and A B in O. Through the points F, N draw G L, H M parallel to A B. Let G L meet the curve of the ellipse again, or the curve of the opposite hyberbola in I, the tangent G H in G, and the tangent L M in L. Let H M meet the curve of the ellipse again, or the curve of the opposite hyperbola in K, the tangent G H in H, and the tangent L M in M. Then, by Prop. XIII. Book I. $BG^2 : IG \times GF :: AL^2 : FL \times LI$; and (34. i.) as B G is equal to A L, and therefore the square of B G equal to the square of A L, we have (14. v.) $IG \times GF$ equal to $FL \times LI$. Consequently, by the sixth Lemma, F G is equal to I L. For the same reasons N H is equal to K M, and (34. i.) as F G is equal to N H, I L is equal to K M; and (34. i.) as L G is equal to M H, I G is equal to K H, and $IG \times GF$ is equal to $KH \times HN$. Again, by Prop. XIII. Book I. $IG \times GF : GB^2 :: KH \times HN : HB^2$; and therefore (14. v.) the square of G B is equal to the square of H B, and G B is equal to H B. Consequently (34. i.) F O is equal to O N.

If, in the ellipse, the straight line D E, meeting the curve in D, E, be parallel to the tangents G H, L M, and if the straight line P D T parallel to A B touch the ellipse in D, then the straight line R E Q parallel to A B will

will touch the ellipſe in E, and D E will be biſected in C, the point in which it meets A B. For let P D T meet the tangent G H in T, and the tangent L M in P; and let R E Q meet the tangent G H in Q, and the tangent L M in R. Then, by Prop. XIII. Book I. $BT^2 : DT^2 :: AP^2 : PD^2$; and (34. i.) as B T, A P are equal; the ſquare of B T is therefore equal to the ſquare of A P, and conſequently (14. v.) the ſquare of D T is equal to the ſquare of D P, and D T is equal to D P. But (34. i.) E Q, D T are equal to one another, as are alſo E R, D P to one another; and therefore E Q, E R are equal to one another. Now if R E Q could meet the curve in any other point beſides E, as in V, then as B Q (34. i.) is equal to A R, it might be proved as above, by means of Prop. XIII. Book I. that V Q × Q E is equal to E R × R V. It would therefore follow, by the ſixth Lemma, that E Q is equal to R V; which by the above is abſurd. Conſequently R E Q touches the ellipſe in E, and therefore, by the Cor. to Prop. XIII. Book I. E Q : Q B :: D T : T B; and as E Q, D T are equal, we have (14. v.) B Q equal to B T. Conſequently C D is equal to C E, for (34. i.) C E is equal to B Q, and C D to B T.

Cor. If two parallel ſtraight lines, as G H, L M touching an ellipſe or oppoſite hyperbolas meet a ſtraight line, as F I, which cuts the ellipſe or oppoſite hyperbolas, and is parallel to the ſtraight line joining the points of contact, the ſegments of the ſecant between the curve or curves and the tangents will be equal to one another; for by the above F G is equal to I L. And if two parallel ſtraight lines touching an ellipſe meet a ſtraight line which touches the ellipſe, and is parallel to the ſtraight line joining the points of contact, the ſegments of the laſt mentioned tangent, between the point of contact and the parallels, will be equal to one another.

:her. This is alfo evident from the above, for it proved that D P is equal to D T. BOOK II.

PROP. II.

wo parallel ftraight lines touch an ellipfe or oppofite yperbolas, and the ftraight line joining the points of conzct be bifected, the point in which it is bifected will be e center of the ellipfe or oppofite hyperbolas; and no her point can be a center of the ellipfe or oppofite hyerbolas.

.et the two parallel ftraight lines G H, L M touch ellipfe A D B E, or oppofite hyperbolas F B N, I A K he points B, A, and let the ftraight line A B, joining points of contact, be bifected in C; the point C is center of the ellipfe or oppofite hyperbolas, and no er point, befides C, can be the center of the ellipfe ppofite hyperbolas. Fig. 37. 38.

'art I. Take any other point N in the curve of the pfe, or in the curve of either of the oppofite hyperas. Then N C being drawn, and produced, it will et the curve of the ellipfe, or the curve of the oppohyperbola, and the whole line terminated by the ve, or curves, will be bifected in C. For let N C be parallel to G H, L M, and draw N O F parallel to m, and let it meet A B in O, and the curve again in Make A S equal to B O, and through S draw I S K allel to the tangents G H, L M, or to N O F, and let e one of the points in which it meets the curve. en, by Prop. I. N F will be bifected in O, and S K l be half the whole line, of which it is a part, terated by the curve; and therefore, by Prop. XIII. ok I. $AO \times OB : NO^2 :: BS \times SA : KS^2$. But as , B O are equal to one another, $AO \times OB$ is equal to $\times SA$, and therefore (14. v.) the fquare of N O is

 equal

BOOK II.

equal to the ſquare of K S, and N O is equal to K S. Let N C produced meet K S I in I, and then as C S, C O are equal, and (29. i.) the triangles N C O, I C S equiangular, it follows (26. i.) that N C is equal to C I, and I S is equal to N O; and therefore, by the above, I S is equal to S K. Conſequently, by Prop. I. the point I is in the curve, and therefore N C being produced, it meets the curve of the ellipſe again, or the curve of the oppoſite hyperbola, and the whole line N I, terminated by the curve, or curves, is biſected in C. The point C is therefore the center of the ellipſe or oppoſite hyperbolas; for in the ellipſe the ſtraight line parallel to the tangents G H, L M, and paſſing through C is alſo biſected in C, by Prop. I.

Part II. No other point, beſides C, can be the center of the ellipſe, or oppoſite hyperbolas. In the ellipſe this is evident; for if there could be another, then a ſtraight line paſſing through C and that other center, and terminated by the curve, would, by the ſecond Definition, be biſected in two points: which is abſurd.

Fig. 38. Nor can the hyperbolas F B N, I A K have any other center beſides C. For through C draw D E parallel to G H, L M; and ſuppoſe the point D in this ſtraight line to be another center. Through D draw the ſtraight line F I parallel to A B, and let it meet the tangents in G, L, and the oppoſite curves in F, I. Then, by Cor. Prop. I. F G is equal to I L; and (34. i.) as G D is equal to B C, and D L to C A, and B C equal to C A, it follows that F D, D I are equal. But through D draw a ſtraight line Q T, not parallel to A B, and let it meet the curves in Q, T; which it evidently may do, by Prop. IX. Book I. as it may be drawn parallel to a ſtraight line drawn from a point in the curve F B N through C and meeting the oppoſite curve, by Part I. Let Q T meet the ſtraight lines F N, I K parallel to the tangents G H,

L M,

L M, in R and P. Let R be between the points F, N, and then P will be without I, K, or without the hyperbola I A K. Then, by the above, as F D is equal to D I, and (29. i.) the triangles F D R, I D P equiangular, it follows (26. i.) that R D is equal to P D. Consequently T Q is not bisected in D, and therefore D is not a center. Nor can any point out of the line D E be a center; for if it could, then a straight line drawn through it, and parallel to A B, and meeting the curves, would, by the above, be bisected by D E, and, according to the first Definition, it would also be bisected in this other center. The same straight line would therefore be bisected in two points: which is absurd. Consequently no other point besides C can be a center. BOOK II.

Cor. 1. A straight line, as A B, joining the points of contact of two parallel tangents G H, L M, is a diameter of the ellipse, or opposite hyperbolas; and D E drawn through the center C, and parallel to the tangents, or to the secants F N, I K parallel to them, is also a diameter. For in the ellipse D E is a diameter, according to the second Definition; and in the hyperbola D E is a second diameter, by the third Definition, as it bisects, by the above, any straight line parallel to A B, and terminated by the curves of the opposite hyperbolas. Fig. 37. 38.

Cor. 2. If two straight lines A L, B G touch an ellipse, or opposite hyperbolas in A, B, the vertices of the diameter A B, they will be parallel. For if B G be not parallel to A L, the tangent parallel to A L will meet the curve of the ellipse, or the curve of the opposite hyperbola, not in B, but in some other point, as N, as two straight lines cannot touch a conic section in the same point. Then, by the preceding Cor. if A N be drawn, it will be a diameter: which is absurd. For the

 straight

BOOK II. ſtraight line A N cannot paſs through C, the center, as C is in the diameter A B.

PROP. III.

If two parallel ſtraight lines touch an ellipſe or oppoſite hyperbolas, ſtraight lines parallel to them in the ellipſe, or in either hyperbola, will be ordinates to the diameter joining the points of contact; but, in the oppoſite hyperbolas, ſtraight lines parallel to the diameter joining the points of contact will be ordinates to the ſecond diameter parallel to the tangents; and, in either caſe, ordinates to the ſame diameter of an ellipſe, or oppoſite hyperbolas, are parallel to one another.

Fig. 37. 38. Part I. Let two parallel ſtraight lines G H, L M touch the ellipſe A D B E, or oppoſite hyperbolas F B N, I A K in the points A, B, and let F N, I K, in the ellipſe or in either hyperbola, be parallel to G H, L M; then A B will be a diameter by Cor. 1. Prop. II. and by Prop. I. it will biſect F N, I K. This part is therefore evident, by the above and the fourth Definition.

Fig. 38. Part II. The reſt remaining as above, let D E be a ſecond diameter, parallel to the tangents G H, L M, and let F I, N K, terminated by the oppoſite curves, be parallel to A B, and then by Cor. 1. Prop. II. F I, N K are biſected by D E, and are therefore ordinates to it, according to the fourth Definition.

Fig. 39. 40. Part III. Ordinates to the ſame diameter of an ellipſe, or oppoſite hyperbolas, are parallel to one another. For firſt let A B be any diameter of the ellipſe, or any tranſverſe diameter of the oppoſite hyperbolas; and C being the center, let L A, G B be the parallel tangents drawn through the vertices A, B, according to Cor. 2. Prop. II. Then, by Part I. any ſtraight line parallel

parallel to L A, or G B in the ellipſe, or in either of the oppoſite hyperbolas, will be an ordinate to A B. But, if it be poſſible, let the ſtraight line I P in the ellipſe, or in either of the oppoſite hyperbolas, be an ordinate to the diameter A B, and not be parallel to L A, G B. Draw I K parallel to L A or G B, and let it meet A B in S, and the curve in K. Draw I C, and, being produced, let it meet the curve of the ellipſe again, or the curve of the oppoſite hyperbola, in N; and draw K N. Then, by Part I. I K is biſected in S; and, as I N is biſected in C, I C : C N : : I S : S K, and therefore (2. vi.) A B, K N are parallel. Conſequently, if I P meet A B in R and K N in V, (2. vi.) I S : S K : : I R : R V, and therefore I V is biſected in R. But the ſtraight line K N in Fig. 39. is wholly within the ellipſe, and in Fig. 40. N K is without the oppoſite hyperbolas, and being produced it falls on the one ſide within one hyperbola, and on the other within the oppoſite hyperbola. The other point P therefore, in which I P meets the curve, cannot be in K N, and conſequently I P cannot be biſected in R, or be an ordinate to A B.

Fig. 40.

Laſtly, the reſt remaining as above, let D E be a ſecond diameter of the hyperbolas parallel to the tangents L A, G B. Then any ſtraight line parallel to A B, and meeting the oppoſite curves will be biſected by D E, according to Part II. But, if it be poſſible, let I X meet the oppoſite curves in I, X, the diameter D E in Y, and be an ordinate to D E, and not be parallel to A B. Let I F be parallel to A B and meet the oppoſite curves in I, F, and D E in D. Draw I C, and, being produced, let it meet the oppoſite curve in N. Draw N F, and let it meet I X in Z. Then, as I N is biſected in C, and as I F, according to Part II. is biſected in D, I C : C N : : I D : D F, and therefore (2. vi.) N F is pa-

 rallel

BOOK II. rallel to D E. Consequently (2. vi.) I D : D F : : I Y : Y Z, and therefore I Y is equal to Y Z. The straight line I X therefore cannot be bisected in Y, or be an ordinate to D E. In every case therefore, ordinates to the same diameter are parallel to one another.

Cor. 1. If a straight line bisect two parallel straight lines in an ellipse or hyperbola, or opposite hyperbolas, it will be a diameter: and straight lines drawn through its vertices, and parallel to the lines bisected, will touch the ellipse, or opposite hyperbolas, if in the opposite hyperbolas it be a transverse diameter. For by Cor. 2. Prop. VIII. Book I. two straight lines, and only two, parallel to the lines bisected, can be drawn to touch the ellipse or opposite hyperbolas; and by Cor. 1. Prop. II. the straight line joining the points of contact is a diameter, and, by this Proposition, this diameter will bisect only such straight lines in the ellipse or opposite hyperbolas as are parallel to the tangents. It is also evident from this Proposition that a straight line bisecting two parallel straight lines, terminated by the curves of opposite hyperbolas, is a second diameter of the hyperbolas.

Cor. 2. If in an ellipse, hyperbola, or opposite hyperbolas, a diameter bisect a straight line not passing through the center, it will also bisect any line parallel to it in the same section or opposite hyperbolas. For, by the preceding Cor. a straight line bisecting two parallel straight lines in an ellipse, hyperbola, or opposite hyperbolas, is a diameter, and therefore passes through the center. Consequently a diameter, or a straight line passing through the center, and bisecting one of two parallel lines in the same section, or opposite hyperbolas, will also bisect the other.

Cor. 3. A diameter of an ellipse or hyperbola will bisect all straight lines in the section parallel to a tangent passing

paſſing through its vertex; and ordinates to a diameter and tangents paſſing through its vertices are parallel to one another. BOOK II.

PROP. IV.

Two diameters of an ellipſe, or oppoſite hyperbolas, are conjugate diameters, if one of them be parallel to the ordinates of the other.

Let A B, D E, be two diameters of the ellipſe A D B E, or of the oppoſite hyperbolas M A, G B F, and let the diameter D E be parallel to G F a double ordinate to A B; the diameter D E will be the conjugate diameter to A B. Fig. 41. 42.

For let C be the center, and F C being drawn, and produced, let it meet the curve of the ellipſe, or the curve of the oppoſite hyperbola in M. Draw G M, and let it meet D E in P. Let A B meet G F in H, and then as G F is a double ordinate to the diameter A B, it is biſected in H; and as the diameter F M is biſected in C, the center, F C : C M : : F H : H G, and therefore (2. vi.) M G is parallel to A B. Again, as D E, G F are parallel, (2. vi.) M C : C F : : M P : P G, and therefore M G is biſected in P. Conſequently the diameters D E, A B are conjugate to one another, according to the ſixth Definition; for by Cor. 2. Prop. III. D E will biſect any ſtraight line parallel to M G or A B, and A B will biſect any ſtraight line parallel to G F or D E, the ſtraight lines parallel to M G or G F being terminated by the curve of the ellipſe, hyperbola, or oppoſite hyperbolas.

Cor. 1. From the above, Cor. 3. Prop. III. and Cor. 1. Prop. II. it is evident, that if two parallel ſtraight lines touch an ellipſe or oppoſite hyperbolas, and if, from any point in the curve of the ellipſe, or of either hyperbola, except the points of contact, a ſtraight

 line

BOOK II. line be drawn parallel to the tangents, it will be either an ordinate to the diameter joining the points of contact, or in the ellipse it will be the diameter conjugate to that joining the points of contact.

Cor. 2. Ordinates to a diameter, of an ellipse or opposite hyperbolas, tangents passing through its vertices, and its conjugate diameter are parallel to one another.

DEFINITIONS.

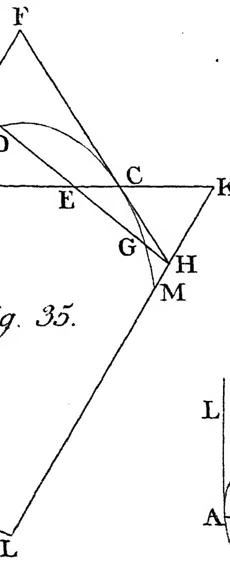
VIII.

Fig. 42. If C be the center of the opposite hyperbolas M A, F B, and A B a transverse diameter, to which D E is the conjugate, and H F an ordinate, and if the rectangle under A H, H B be to the square of H F as the square of C B to the square of C E or C D, the points D, E are called *the Vertices of the second Diameter* D E. In this way the magnitude of any second diameter is determined by its vertices.

Cor. If the ordinate H F be parallel to the base of the cone, in which the hyperbola was formed, the semidiameter C E or C D will be equal to the straight line drawn through C and parallel to the base of the cone, and touching either of the conical superficies. For, in this case, by the second Cor. to Prop. X. Book I. the rectangle under A H, H B is to the square of H F as the square of C B to the square of the line drawn through C and parallel to the base of the cone, and touching either of the conical superficies; and, by this Definition, the square of C B is to the square of C E or D E in the same proportion. Consequently (9. v.) the Cor. is evident, as the sides of equal squares are equal.

IX.

A straight line which is a third proportional to two conjugate diameters of an ellipse, or opposite hyperbolas,



g. 35.

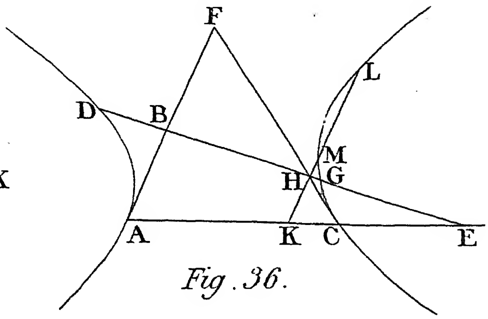


Fig. 36.

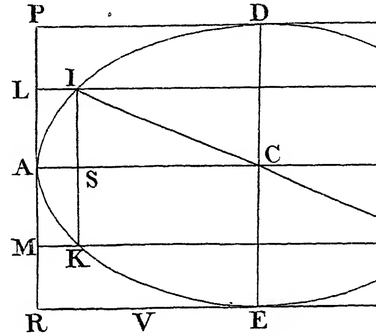


Fig. 37.

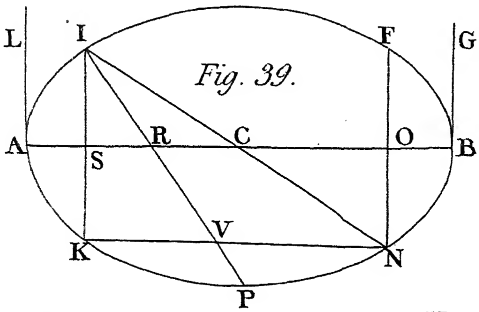


Fig. 39.

Fig. 38.

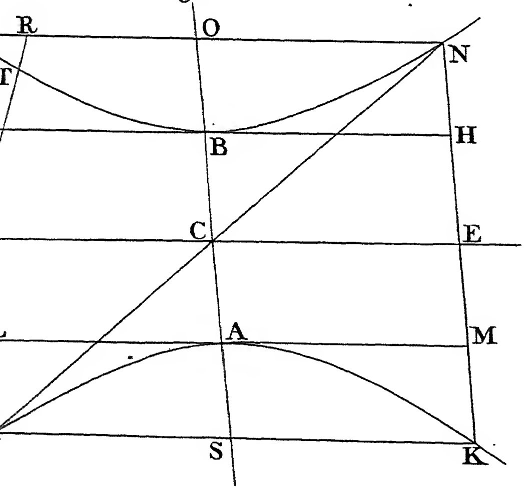
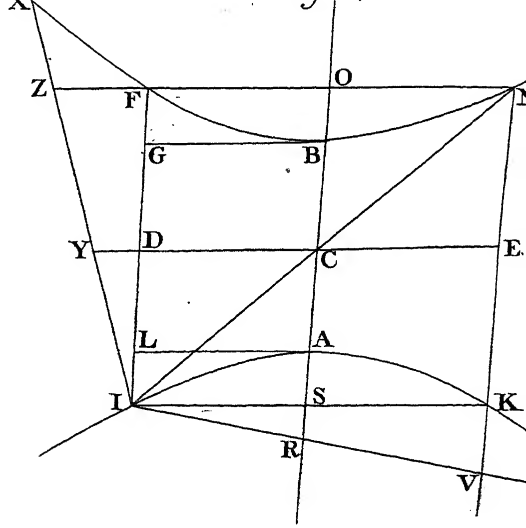


Fig. 40.



If each of two straight lines, meeting one another, touch or cut, or one of them touch, and the other cut, an ellipse, hyperbola, or opposite hyperbolas; the square of the first of the two, if a tangent, or the rectangle under its segments, if a secant, will be to the square of the second, if a tangent, or the rectangle under its segments, if a secant, as the square of the semidiameter parallel to the first to the square of the semidiameter parallel to the second.

In the ellipse, and when the two straight lines are parallel to two transverse diameters of opposite hyperbolas, the Proposition is evident from Prop. XIII. Book I. For diameters in the ellipse, and transverse diameters of opposite hyperbolas, are secants meeting one another in the center, in which they are bisected.

For other cases, first let L M, G H cutting either of the opposite hyperbolas in L, M and G, H, meet one another in K; and let C K be the semidiameter parallel to L M, and let C F be the semidiameter parallel to G H, and let C E, L M be parallel to the base of the cone in which the hyperbola was formed; and then $LK \times KM : GK \times KH :: CE^2 : CF^2$. For, bisect G H in I, and through I draw N P parallel to L M or C E, and draw the transverse diameter I B A. Then, by Cor. 2. to Prop. X. Book I. and the Cor. to Definition VIII. $NI \times IP : AI \times IB :: CK^2 : CB^2$; and by Definition VIII. $AI \times IB : IH^2 :: CB^2 : CF^2$. Consequently, Fig. 45.

$$\begin{array}{ccccc} NI \times IP & : & AI \times IB & : & IH^2 \\ CE^2 & : & CB^2 & : & CF^2, \end{array}$$

and (22. V.) $NI \times IP : IH^2 :: CE^2 : CF^2$. But, by Prop.

BOOK II. Prop. XIII. Book I. N I × I P : I H^2 : : L K × K M : G K × K H; and therefore (11. v.) L K × K M : G K × K H : : C E^2 : C F^2.

Fig. 44. Secondly, let G H, L M cut one another in K, in the hyperbola as above, and let neither of them be parallel to the baſe of the cone in which the ſection was formed. Let G H be parallel to the ſemidiameter C F, and let L M be parallel to the ſemidiameter C E. Through K draw Q R parallel to the baſe of the cone in which the ſection was formed, and let C S be the ſemidiameter parallel to Q R, or to the baſe of the cone. Then, as above, we have the two following ranks of magnitudes proportionals,

L K × K M : Q K × K R : G K × K H
C E^2 : C S^2 : C F^2;

and therefore (22. v.) L K × K M : G K × K H : : C E^2 : C F^2.

Thirdly, the reſt remaining as above, let the ſtraight line V T K meet one of the oppoſite hyperbolas in V, the other in T, and the ſtraight line Q R in K. Then, by Prop. XII. Book I. and Cor. to Definition VIII. Y X being the tranſverſe diameter parallel to V T, V K × K T : Q K × K R : : C X^2 : C S^2.

Laſtly, every thing remaining as in the two preceding caſes, by the above we have the two following ranks of magnitudes proportionals,

V K × K T : Q K × K R : G K × K H
C X^2 : C S^2 : C F^2, and therefore (22. v.) V K × K T : G K × K H : : C X^2 : C F^2.

In every caſe therefore, by the above, and Prop. XIII. Book I. (and 11. v.) "If each of two ſtraight lines," &c.

Cor. 1. If two ſtraight lines be ordinates to any diameter of an ellipſe, or tranſverſe diameter of an hyperbola, the ſquare of the firſt will be to the ſquare of the

ſecond,

econd, as the rectangle under the abſciſſes correſponding to the firſt to the rectangle under the abſciſſes correſponding to the ſecond.

Cor. 2. From this Propoſition (and 22. vi.) it is evident, that if two ſtraight lines meeting one another touch an ellipſe, hyperbola, or oppoſite hyperbolas, they will be to one another as the ſemidiameters to which they are parallel.

Cor. 3. From this Propoſition, and the firſt Lemma, it is evident, that if two conjugate diameters of an ellipſe cut one another at right angles, they cannot be equal to one another; for if they were equal to one another, the ſection would be a circle, by the firſt Lemma, as the ſquare of the ordinate would be equal to the rectangle under the correſponding abſciſſes.

PROP. VI.

If a ſtraight line be an ordinate to any diameter of an ellipſe, or any tranſverſe diameter of an hyperbola, the rectangle under the abſciſſes of the diameter will be to the ſquare of the ordinate as the diameter to its parameter.

Let the ſtraight line G F be an ordinate to the diameter A B of the ellipſe B G, or to the tranſverſe diameter A B of the hyperbola B G, and let B H be the parameter of A B; the rectangle under the abſciſſes A F, F B is to the ſquare of F G as A B to B H. Fig. 45. 46.

Let C be the center of the ellipſe or hyperbola, and let D E be the diameter parallel to G F, and conſequently, by Prop. IV. the conjugate diameter to A B. Then, by the ninth Definition, A B : D E : : D E : B H; and therefore (Cor. 2. 20. vi.) $AB^2 : DE^2 :: AB : BH$. But (15. v.) $AB^2 : DE^2 :: CB^2 : CD^2$; and therefore (11. v.) $CB^2 : CD^2 :: AB : BH$. But, by Prop. V.

CB^2

BOOK II. $CB^2 : CD^2 :: AF \times FB : FG^2$; and consequently (11. v.) $AF \times FB : FG^2 :: AB : BH$.

Cor. 1. Let the parameter B H be at right angles to the diameter A B, and from the other vertex A, draw A H. From the point F draw F K perpendicular to A B, and let it meet A H, or A H produced, in K. Complete the rectangle K B, and it will be equal to the square of the ordinate F G. For, as B H, F K are at right angles to A B, they are parallel, and therefore (4. vi.) $AB : BH :: AF : FK$; and consequently, by this Proposition (and 11. v.) $AF \times FB : FG^2 :: AF : FK$. But (1. vi.) $AF : FK :: AF \times FB : FK \times FB$, and therefore $AF \times FB : FG^2 :: AF \times FB : FK \times FB$. Consequently (14. v.) $FK \times FB$ is equal to FG^2.

Cor. 2. Complete the rectangle L A B H, and let L H meet F K in M, and let K N, the side of the rectangle K B, opposite to B F, meet B H in N; then in the ellipse the square of the ordinate F G is less than the rectangle under the abscis F B and the parameter B H, by the rectangle M N, similar to L B, and having one of its sides equal to B F; but in the hyperbola, the square of the ordinate F G, is greater than the rectangle under the abscis B F and the parameter B H, by the rectangle M N similar to L B, and having one of its sides equal to B F. This is evident from the preceding Cor.

SCHOLIUM.

On account of the deficiency of the square of F G from the rectangle under F B, B H in Fig. 45. Apollonius called the section an ellipse; and on account of the excess of the square of F G above the rectangle under F B, B H in Fig. 46. he called the section an hyperbola.

From the properties demonstrated above these sections

Fig. 45. and 46.) $= a$, its parameter B H $= p$, the abfcifs F B $= x$, and the ordinate F G $= y$. Then A F $=$ $a \mp x$, the negative fign applying to the ellipfe, and the pofitive fign to the hyperbola. And by the fimilar triangles A B H, A F K, $a : p :: a \mp x : \frac{ap \mp px}{a} =$ FK.

Confequently by the firft Cor. to Prop. VI. $\frac{ap \mp px}{a} \times x$ $= px \mp \frac{px^2}{a} = px \mp \frac{p}{a}x^2 = y^2$.

PROP. VII.

If a ftraight line, touching an ellipfe or hyperbola, meet a diameter, and from the point of contact there be drawn an ordinate to the diameter; the femidiameter will be a mean proportional between the fegments of the diameter, between the center and ordinate, and between the center and tangent.

Firft, let the ftraight line E M, touching the ellipfe or hyperbola E I G in the point E, meet any diameter A I in the ellipfe or tranfverfe diameter of the hyperbola in M, and let E F be an ordinate to the diameter, and meet it in F, and let C be the center; the femidiameter C I is a mean proportional between the fegments C F, C M. Fig. 47. 48.

For, let A B, I D be tangents paffing through the vertices A, I, and meeting the tangent E M in B and D. Then, by Cor. to Prop. XIII. Book I. E B : E D :: A B : I D; and as, by Cor. 3. Prop. III. the ftraight lines A B, E F, I D are parallel, it is evident (from 10. vi.) that E B : E D :: A F : I F. And as (29. i.) the triangles B A M, D I M are equiangular, A B : I D ::

A M

BOOK II. A M : I M. Hence (11. v.) A M : I M : : A F : I F, and (18. v.) A M + I M : I M : : A F + I F : I F; and by halving the antecedents it will be in the ellipse C M : I M : : C I : I F, but in the hyperbola C I : I M : : C F : I F. Consequently, by conversion, it will be in the ellipse C M : C I : : C I : C F; but in the hyperbola C I : C M : : C F : C I, and therefore C M : C I : : C I : C F.

Fig. 49. Let A I be now a second diameter of the opposite hyperbolas G K, E L and E F an ordinate to it; and let the tangent E M meet it in M, and the transverse diameter K L, parallel to E F, in N. Then, by Prop. IV. K L, A I are conjugate diameters; and therefore E P being drawn parallel to A I, and meeting K L in P, it will be an ordinate to K L. Let C be the center, and then, by the above, C P : C L : : C L : C N, and therefore (Cor. 2. 20. vi.) C P^2 : C L^2 : : C P : C N. But (34. i.) C P, F E are equal, and E P is equal to F C, and (4. vi.) E F : C N : : M F : M C, and therefore C P^2 : C L^2 : : M F : M C. Hence (17. v. and 6. ii.) K P × P L : C L^2 : : C F : M C; and (1. vi. and 11. v.) K P × P L : C L^2 : : C F^2 : C F × M C; and (16. v.) K P × P L : C F^2 or E P^2 : : C L^2 : C F × M C. But, by Def. viii. K P × P L : E P^2 : : C L^2 : C I^2, and therefore (11. and 9. v.) C F × M C is equal to C I^2, and consequently C M : C I : : C I : C F.

Fig. 47. 48. *Cor.* 1. From the above (and 17. vi.) C M × C F is equal to C I^2, and in the ellipse these equals being taken from C M^2, we have (6. ii. and 2. ii.) A M × M I equal to C M × M F. But, when A I is a transverse diameter in the hyperbola, C M^2 being taken from the equals, C M × C F, C I^2, we have (5. ii. and 3. ii.) A M × M I equal to C M × M F.

Cor. 2. As C M × C F is equal to C I^2, by taking from each C F^2 in the ellipse, we have (3. and 5. ii.)

C F

C F × F M equal to A F × F I. But A I being a tranſverſe diameter in the hyperbola, by taking the equals C M × C F, CI^2 from CF^2, we have (2. and 6. ii.) C F × F M equal to A F × F I.

Cor. 3. When A I is a diameter of the ellipſe or tranſverſe diameter of the hyperbola, by the demonſtration of the firſt part of the Propoſition, A M : I M : : A F : F I.

PROP. VIII.

If two ſtraight lines, touching an ellipſe, hyperbola, or oppoſite hyperbolas, meet one another, the diameter biſecting the line joining the points of contact will paſs through the point of concourſe.

Let the two ſtraight lines E M, G M touch the ellipſe or hyperbola E I G, or the oppoſite hyperbolas E L, G K, in the points E, G, and meet one another in M, and let the diameter C F biſect E G, the ſtraight line joining the points of contact in F; the diameter C F will paſs through M. Fig. 47. 48. 49.

For let C be the center, and let A, I be the vertices of the diameter; and then, as E G is biſected by the diameter C F, E F is an ordinate to it, and therefore, by Prop. VII. C F : C I : : C I : the ſegment of the diameter intercepted between C and the tangent E M. For the ſame reaſons C F : C I : : C I : the ſegment of the diameter intercepted between C and the tangent G M. Conſequently the ſegment of the diameter between C and the tangent E M, is equal to the ſegment of the diameter between C and the tangent G M. The diameter muſt therefore paſs through M; for if it did not, it would, upon being produced, firſt meet the one tangent, and then the other, and its ſegments between C and the tangents would be unequal.

Cor.

ing the line joining the points of contact, will be a diameter.

PROP. IX.

If two parallel straight lines touching an ellipse, or opposite hyperbolas, meet a third tangent, the rectangle under their segments, between the points of contact and the points of concourse, will be equal to the square of the semidiameter to which they are parallel; and the rectangle under the segments of the third tangent, between its point of contact and the parallel tangents, will be equal to the square of the semidiameter to which it is parallel.

Fig. 50. 51. 52. Let the parallel straight lines A B, I D touch an ellipse E I, or opposite hyperbolas A, I, in the points A, I, and let them meet in B, D a straight line B D, which touches the ellipse, or one of the opposite hyperbolas in E; then the rectangle under the segments A B, I D is equal to the square of the semidiameter parallel to A B or I D, and the rectangle under the segments B E, E D is equal to the square of the semidiameter parallel to B D.

For let C be the center, and draw E G parallel to A B, I D; and draw also A I. Then by Cor. 1. to Prop. II. A I is a diameter, and, by Cor. 1. to Prop. IV. E G is either an ordinate to A I, or in the ellipse the conjugate diameter to it.

Fig. 50. First, let E G be the conjugate diameter to A I, and then by Prop. IV. and Cor. 3. Prop. III. B D, A I are parallel to one another, and also A B, C E, I D to one another. Consequently (34. i.) A B, C E, I D are equal to one another, as are also A C, C I, B E, E D to one another;

Fig. 41.

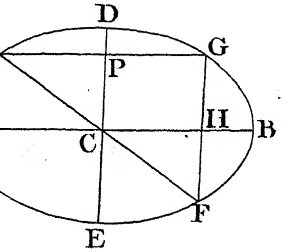


Fig. 42.

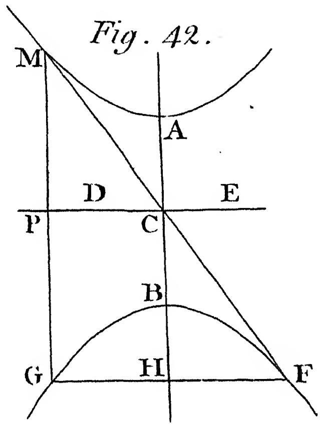


Fig. 43.

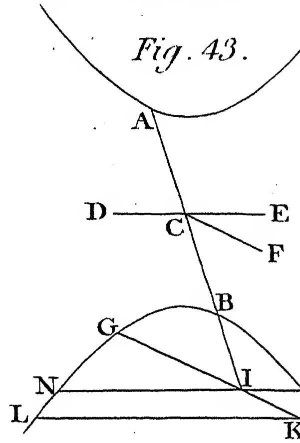


Fig. 44.

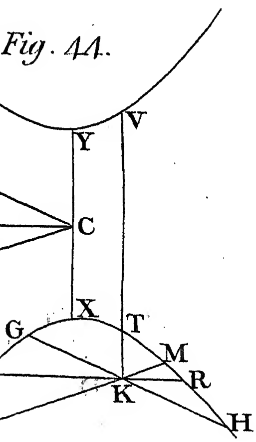


Fig. 45.

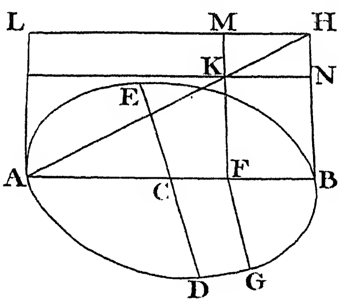


Fig. 46.

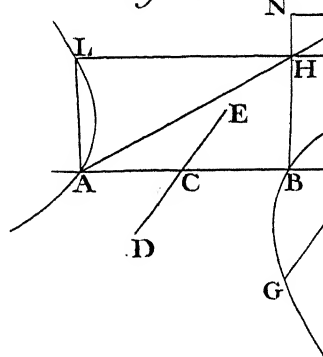


Fig. 47.

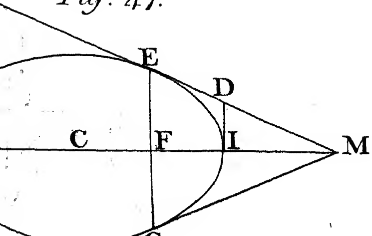


Fig. 48.

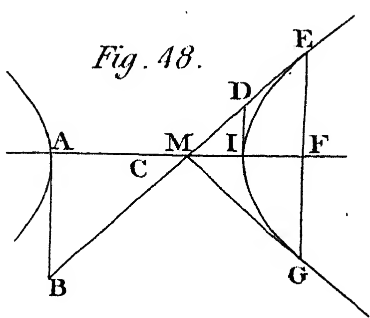
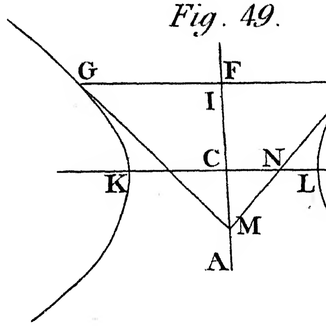


Fig. 49.



another; and therefore A B × I D is equal to C E², and B E × E D is equal to A C². BOOK II.

Next, let E G be an ordinate to the diameter A I, and let it meet it in F. Let C K be the femidiameter parallel to the tangents A B, I D, and C P the femidiameter parallel to the tangent E B; and let E B meet the diameter A I in M, and the diameter K C L in H. Let E O be an ordinate to K C L, and let it meet it in O. Then, by Cor. 3. Prop. III. and Prop. IV. A B, L H, G E, I D are parallel, and E O is parallel to A I. By Cor. 1. Prop. VII. A M × M I is equal to C M × M F, and therefore (16. vi.) A M : C M : : M F : M I; and (4. vi.) A M : C M : : A B : C H, and M F : M I : : F E or C O : I D. Confequently (11. v.) A B : C H : : C O : I D, and (16. vi.) A B × I D is equal to C H × C O, which is equal to C K², by Prop. VII. (and 17. vi.) and therefore A B × I D is equal to C K². Fig. 51. 52.

Again, by Cor. 2. Prop. V. A B : B E : : C K : C P, and by Cor. to Prop. XIII. Book I. A B : I D : : B E : E D. Hence (22. vi.) A B × I D : B E × E D : : C K² : C P², and as by the above A B × I D is equal to C K², it follows (14. v.) that B E × E D is equal to C P².

Cor. 1. Every thing remaining as above, as, by Cor. 2. Prop. VII. A F × F I is equal to C F × F M, (16. vi.) A F : F M : : C F : F I. But on account of the parallels (10. vi.) A F : F M : : B E : E M; and C F : F I : : H E : E D. Confequently (11. v.) B E : E M : : H E : E D, and therefore (16. vi.) M E × E H is equal to B E × E D or to C P².

Cor. 2. If a ftraight line as B E, touching an ellipfe or hyperbola in the point E, meet the diameter A I in M, and the diameter K L in H, and if the rectangle under M E, E H be equal to the fquare of the femidiameter parallel to B E, the diameters A I, K L will be conjugate to one another.

F PROP.

BOOK II.

PROP. X.

To find the axes of a given ellipſe or hyperbola, the center being alſo given; and to demonſtrate that the ſame ſection can have only two axes.

Fig. 53. 54. Part I. Let D B E be a given ellipſe or hyperbola, of which C the center is alſo given; it is required to find the axes of the ſection.

Fig. 53. In the ellipſe draw C G, C F two ſemidiameters, and, if they be unequal, let C G be greater than C F. With C as a center, and a diſtance leſs than C G but greater than C F, deſcribe the circle H D E. Then from this conſtruction and the nature of the two curves it is evident, that the circumference of the circle will cut the curve of the ellipſe in four points, two of them being towards the left of the center, as the figure is viewed, and two of them towards the right. Let the circumference of the circle cut the curve of the ellipſe in the points D, E. Draw the ſtraight line D E; and through C draw A B biſecting D E in I, and (3. iii.) A B will be at right angles to D E. Through C draw L M parallel to D E, and A B, L M will be the axes of the ellipſe.

For, by conſtruction, D E is an ordinate to the diameter A B, and at right angles to it. Again, as L M is parallel to D E, and as A I D is a right angle, the angle A C L (29. i.) is alſo a right one. By Prop. IV. the diameter L M will alſo biſect all ſtraight lines in the ellipſe parallel to A B, and it will therefore cut its ordinates at right angles. Conſequently A B, L M are axes of the ellipſe, according to the ſeventh Definition.

If the ſemidiameters C G, C F be equal, then a diameter biſecting the angle G C F will be one of the axes, and a diameter at right angles to it will be the other. For, in this caſe, if G F be drawn, it will be biſected at right

right angles (4. i.) by the diameter bifecting the angle G C F; and the reft will be as above. BOOK II.

Next, let the fection D B E be an hyperbola, of which C is the center, and let K be any point within the hyperbola. With C as a center, and C K as a diftance, defcribe the circle E K D, and let its circumference cut the curve of the hyperbola in the points E, D. Draw D E and bifect it in I, and through I draw the diameter A B; and parallel to D E draw the diameter L M. The diameters A B, L M are the axes of the hyperbola. For D E is a double ordinate to the diameter A B, and (3. iii.) A B cuts it at right angles, and L M is parallel to the ordinate D E. Fig. 54.

Part II. To demonftrate that an ellipfe or hyperbola can have only two axes. Firft, let the fection B D A be an ellipfe, and C being the center, let R L, F G be the axes, found as above; and, if it be poffible, let the diameter A B be alfo an axis. Let L D be a double ordinate to A B, meeting it in E, and the curve again in D; and the diameter D K being drawn, let D H be an ordinate to R L*. Then, as by hypothefis A B is an axis, C E L, C E D are right angles, and as L D is bifected in E, (4. i.) C L is equal to C D. Again, as by the above R L, F G are conjugate, D H is parallel to F G, by Prop. IV. and by Prop. V. $CL^2 : CF^2 :: RH \times HL : DH^2$. But, by Cor. 3. Prop. V. C L, C F muft be unequal, and therefore, fuppofing C L to be the greateft, CL^2 is greater than CF^2, and $RH \times HL$ greater than DH^2. To thefe unequals add CH^2, and then (5. ii. and 47. i.) CL^2 is greater than CD^2; and confequently C L is greater than C D. But C D is alfo equal to C L: which Fig. 55.

* A method of drawing an ordinate to a given diameter will be inferted hereafter. The infertion of it previous to the above, or at this place, would have caufed a needlefs repetition.

 is

BOOK II. is abſurd. The diameter A B therefore cannot be an axis.

Fig. 56. Next, let the ſection E B D be an hyperbola, of which C is the center, F A the oppoſite hyperbola, and A B, L M the axes found as above; and, if it be poſſible, let the tranſverſe diameter D F be an axis. Let D T touch the hyperbola in D, and meet the axis A B in T; and let D I E be an ordinate to A B, and let it meet it in I. Then in the triangle C I D, C I D is a right angle, by Def. VII. and therefore C D I is leſs than a right angle, and conſequently C D T is much leſs than a right angle. But, by Cor. 2. Prop. IV. the tangent T D is parallel to the ordinates of the axis F D, and therefore, by Def. VII. (and 29. i.) the angle T D C is a right one. And, by the above, it is alſo leſs than a right one: which is abſurd. Conſequently D F is not an axis. Nor can a ſecond diameter as G H, beſides L M, be an axis. For F D conjugate to G H being drawn, and D T a tangent, the demonſtration would end in the ſame abſurdity.

Cor. From the above, and Prop. IV. it is evident, that the axes of an ellipſe, or hyperbola, are conjugate diameters.

PROP. XI.

Of all the diameters of an ellipſe the greater axis is the greateſt and the leſſer axis is the leaſt; and of oppoſite hyperbolas the axes are the leaſt diameters.

Fig. 55. Part I. Let A B D be an ellipſe, of which C is the center, R L the greater axis, and F G the leſſer axis; of all the diameters R L is the greateſt and F G the leaſt.

For let K D be any other diameter, and let D H be an ordinate to R L, and D M an ordinate to F G. Then, by Cor. 2. Prop. IV. D H is parallel to F G, and D M to

R L;

R L; and therefore, by Prop. V. $CL^2 : CF^2 :: RH \times HL : DH^2$, and as C L is greater than C F, CL^2 is greater than CF^2, and $RH \times HL$ is greater than DH^2. To these add CH^2, and (5. ii. and 47. i.) then CL^2 is greater than CD^2. Consequently C L is greater than C D, and therefore R L is greater than K D. Again, by Prop. V. $CL^2 : CF^2 :: DM^2 : GM \times MF$, and therefore, as above, DM^2 is greater than $GM \times MF$. To these add the square of C M, and then (5. ii. and 47. i.) CD^2 is greater than CF^2. Consequently K D is greater than F G.

Part II. Let A B, L M be the axes of the opposite hyperbolas E B D, A F, and F D, G H any other conjugate diameters; then A B is less than the transverse diameter F D, and L M is less than G H. Fig. 56.

For let D E be an ordinate to the axis A B, and let it meet it in I, and let D T touch the hyperbola in D, and let it meet A B in T. Let B P touch the hyperbola in the vertex B, and let it meet the tangent D T in P, and let C be the center. Then, by Def. VII. C I D is a right angle, and therefore (19. i.) C D is greater than C I, and consequently much greater than C B. Hence the transverse axis is less than the transverse diameter F D. Again, by Cor. 2. Prop. IV. G H is parallel to T D, and D I to P B; and, by Prop. VII. $CI : CB :: CB : CT$, and therefore by conversion, $CI : BI :: CB : BT$. Hence, as C I is greater than C B, (14. v.) B I is greater than B T. But (2. vi.) $BI : BT :: DP : PT$, and therefore D P is greater than P T; and, as (29. i.) P B T is a right angle, P T (19. i.) is greater than B P. Hence D P is greater than B P; and as, by Cor. 2. Prop. V. $DP : BP :: CH : CL$, C H is greater (14. v.) than C L. Consequently the axis L M is less than the second diameter G H.

BOOK II.

DEFINITIONS.

X.

Fig. 60. 61. In the ellipse the greater axis is called the *Transverse Axis*, and the other axis is called the *Conjugate Axis*; and in the hyperbola the axis which is a transverse diameter is called the *Transverse Axis*, and the other axis is called the *Conjugate Axis*.

XI.

If C be the center, A B the transverse axis, and D E the conjugate axis of the ellipse A D B, or of the opposite hyperbolas A I, B P, then if in A B two points F, O be so taken that the rectangle under A F, F B, and also the rectangle under A O, O B, be equal to the square of C D or C E, the semiconjugate axis; the points F, O are called the *Foci*, or *Umbilici*, of the ellipse, hyperbola, or opposite hyperbolas.

Cor. 1. As (axiom i. 1.) A F × F B is equal to A O × O B, the foci F, O are equally distant from the vertices A, B, by the sixth Lemma. It is also evident, that the foci are equally distant from the center.

Fig. 60. *Cor.* 2. In the ellipse the distance of each of the foci from either extremity of the conjugate axis is equal to the semitransverse axis. For, supposing a straight line to be drawn from D to O, the square of D O (47. i.) will be equal to the squares of C O, C D together; and therefore, by this Definition, (and 5. ii.) the square of D O is equal to the square of A C. Consequently D O is equal to A C; and therefore if with D or E as a center, and A C or C B as a distance, a circle be described, the circumference will cut A B in O, and F the foci.

Fig. 61. *Cor.* 3. In the hyperbola the distance of each of the foci from the center is equal to the distance between the vertices of the transverse and conjugate axes. For, supposing a straight line to be drawn from D to A, the square of D A (47. i.) will be equal to the squares of

C A,

C A, C D together; and therefore, by this Definition, (and 6. ii.) the ſquare of D A is equal to the ſquare of C O or C F. Conſequently C O or C F is equal to D A; and therefore the foci F, O may be eaſily found from the axes. BOOK II.

Cor. 4. The double ordinate T S to the axis A B, drawn through either focus, ſuppoſe F, is equal to the parameter of the axis A B. For, by Prop. V. $CB^2 : CD^2 :: AF \times FB$, or by this Def. as $CD^2 : TF^2$; and therefore (22. vi.) $CB : CD :: CD : TF$. Conſequently (15. 5.) $AB : DE :: DE : TS$, and the Cor. is evident from Def. IX. Fig. 60. 61.

XII.

If through the foci F, O of an ellipſe A D B, or of the oppoſite hyperbolas A I, B P, ordinates F T, O I to the axis A B be drawn, and if through the points T, I in which they meet the curve ſtraight lines H T, I L be drawn to touch the ſection, or oppoſite hyperbolas, the tangents H T, I L are called *Focal Tangents.*

PROP. XII.

If a tangent, paſſing through a vertex of the tranſverſe axis of an ellipſe or oppoſite hyperbolas, meet a focal tangent, its ſegment between the point of contact and point of concourſe will be equal to the ſegment of the axis between the point of contact and the focus, to which the focal tangent belongs.

Let T B be an ellipſe or hyperbola, of which A B is the tranſverſe axis, and F, O the foci, and let A H touch the ellipſe, or either of the oppoſite hyperbolas, in the vertex A, and meet in the point H the focal tangent H G, belonging to the focus F; the ſegment A H is equal to the ſegment A F. Fig. 60. 61.

For let H G touch the ſection in T, and T F, being

 drawn,

BOOK II. drawn, will be an ordinate to A B, by the twelfth Definition. Let B G touch the section in the vertex B, and meet H G in G; and let C be the center, and D E the conjugate axis. Then, by Cor. 2. Prop. IV. A H, D E, T F, B G are parallel; and therefore, by Cor. to Prop. XIII. Book I. A H : B G : : H T : T G. But it is evident, (from 10. vi.) that H T : T G : : A F : F B; and therefore (11. v.) A H : B G : : A F : F B. Consequently A H × B G is similar to A F × F B, and, by Prop. IX. and Def. XI. each of these rectangles is equal to the square of C D. They are therefore equal to one another; and, as they are also similar, A H is equal to A F, and B G is equal to B F.

Cor. If O I be drawn an ordinate to A B, and on the side of A B opposite to that on which F T is, and if through I, the point in which it meets the curve, there be drawn the focal tangent K L, meeting the tangent H A in K, and the tangent G B in L; then H L will be a parallelogram, and each of the opposite sides H K, G L will be equal to the transverse axis A B. For, by the above, and Cor. 1. to Def. XI. A K, A O, B F, B G are equal to one another, and also A H, A F, B O, B L to one another. Consequently H K, G L are equal and parallel, and therefore (33. i.) H G, K L are equal and parallel. Hence H L is a parallelogram; and as A H is equal to A F, and A K to B F, H K or G L is equal to A B.

PROP. XIII.

If from any point in the curve of an ellipse or hyperbola two straight lines be drawn to the foci, their sum in the ellipse, but their difference in the hyperbola, will be equal to the transverse axis.

Fig. 60. 61. Let P be any point in the curve of the ellipse, or hyperbola P T B, of which A B is the transverse axis, and

and the points F, O the foci, and let P O, P F, be ſtraight lines drawn to the foci; the ſum of P O, P F in the ellipſe, but their difference in the hyperbola, is equal to A B.

For, the reſt remaining as in the preceding Propoſition and its Corollary, let P R be drawn an ordinate to A B, and let it meet the curve again in M, the focal tangent G H in N, and the focal tangent K L in Q. Then, by Cor. 2. Prop. IV. H K, O I, D E, N Q, T F, G L are parallel, and therefore, by Prop. XIII. Book I. $TH^2 : TN^2 :: AH^2 : MN \times NP$. But, on account of the parallels (10. vi.) $TH^2 : TN^2 :: FA^2 : FR^2$; and therefore (11. v.) $FA^2 : FR^2 :: AH^2 : MN \times NP$, and as, by Prop. XII. F A is equal to A H, FA^2 is equal to AH^2, and therefore FR^2 is equal to $MN \times NP$. To theſe equals add the ſquare of P R, and then (6. ii. and 47. i.) the ſquare of N R is equal to the ſquare of P F, and conſequently N R is equal to P F. Again, by Prop. XIII. Book I. $IL^2 : IQ^2 :: BL^2 : PQ \times QM$; and on account of the parallels (10. vi.) $IL^2 : IQ^2 :: OB^2 : OR^2$, and therefore (11. v.) $OB^2 : OR^2 :: BL^2 : PQ \times QM$. But, by Prop. XII. O B is equal to B L, and therefore OB^2 is equal to BL^2, and (14. v.) OR^2 is equal to $PQ \times QM$. To theſe equals add the ſquare of R M, or its equal the ſquare of R P, and then (6. ii. and 47. i.) the ſquare of O M in the ellipſe, and the ſquare of O P in the hyperbola, is equal to the ſquare of R Q. Conſequently in the hyperbola O P is equal to R Q, and in the ellipſe O M is equal to R Q. But in the ellipſe P R, R M are equal, and the angles O R M, O R P are equal, being right angles, and O R is common to the two triangles O R M, O R P, and therefore (4. i.) O M is equal to O P. In each ſection therefore O P is equal to R Q, and P F to N R. Conſequently in the ellipſe the ſum of P O, P F, but in the hyperbola their dif-

BOOK II. difference is equal to N Q. But Q H is a parallelogram, and therefore (34. i.) N Q, H K are equal; and as, by the Cor. to Prop. XII. H K is equal to A B, the sum of P O, P F in the ellipse, but their difference in the hyperbola is equal to A B, the transverse axis.

Cor. 1. If from any point in the curve of an ellipse, or hyperbola, two straight lines be drawn to the foci, in the ellipse the difference between the transverse axis and either of the two will be equal to the other; but in the hyperbola the sum of the transverse axis and the least of the two will be equal to the other.

Cor. 2. If the conjugate axis D E be produced till it meet the opposite focal tangents in V and W, each of the segments C V, C W between C the center and a focal tangent will be equal to the semitransverse axis. For, let the opposite focal tangents meet the transverse axis A B in X and Y. Then, as, by Cor. 1. Def. XI. C F, C O are equal, it is evident, from Prop. VII. that C Y, C X are equal; and as X W, V Y are parallel, the angles (29. i.) C X W, C Y V are equal. Consequently, as the angles at C are right angles, (26. i.) C V is equal to C W. In each section therefore the Cor. is evident.

PROP. XIV.

If from a point without an ellipse or opposite hyperbolas two straight lines be drawn to the foci, their sum in the ellipse will be greater, but their difference in the hyperbola will be less, than the transverse axis. But if from a point within an ellipse or hyperbola two straight lines be drawn to the foci, their sum in the ellipse will be less, but their difference in the hyperbola greater, than the transverse axis.

Fig. 57. 58. Part I. Let E be a point without the ellipse A D B, or opposite hyperbolas A, B, and let E F, E O be straight lines

lines drawn to the foci F, O; the ſum of E F, E O in the ellipſe is greater, but their difference in the hyperbolas leſs, than A B the tranſverſe axis. BOOK II.

In the ellipſe let E F cut the curve in D, and draw O D; and then (20. i.) O E, E D together being greater than O D, the three O E, E D, D F together are greater than O D, D F together. Conſequently, by Prop. XIII. the ſum of E F, E O is greater than A B. In the hyperbolas let E F be greater than E O, and let E O cut the curve of the hyperbola A in D, and draw F D. Then D F, D E together are greater than E F. But, by the Cor. to Prop. XIII. D F is equal to A B, O D together, and therefore A B, O D, D E together, or A B and O E together, are greater than E F. Conſequently the difference between E F, E O is leſs than A B. Fig. 57. Fig. 58.

Part II. Let G be a point within the ellipſe or hyperbola A D, and let the ſtraight lines G F, G O be drawn to the foci F, O; the ſum of G F, G O in the ellipſe is leſs, but in the hyperbola their difference is greater, than A B the tranſverſe axis.

In either ſection let G F meet the curve in D, and draw D O. Then, in the ellipſe, O D, D G together (20. i.) are greater than O G; and therefore O D, D G, G F together, or O D, D F together, are greater than O G, G F together. Conſequently the ſum of G F, G O is leſs than A B, by Prop. XIII. In the hyperbola O D, D G together (20. i.) are greater than G O, and therefore O D, D G, A B together are greater than G O, A B together. But, by Cor. 1. to Prop. XIII. O D, A B together are equal to D F, and therefore D F, D G together, or F G, are greater than G O, A B together. Conſequently the difference between F G, G O is greater than A B.

Cor. 1. From this and Prop. XIII. it is evident, that two ſtraight lines being drawn from a point to the foci of

BOOK II. of an ellipſe or hyperbola, if in the ellipſe their ſum be greater, or in the hyperbola their difference be leſs, than the tranſverſe axis, the point will be without the ſection. If in the ellipſe the ſum of the two lines, or in the hyperbola their difference, be equal to the tranſverſe axis, the point will be in the curve of the ſection. Laſtly, if in the ellipſe the ſum of the two lines be leſs, or in the hyperbola their difference be greater, than the tranſverſe axis, the point will be within the ſection.

Fig. 59. *Cor.* 2. If O, F be the foci of the hyperbola B P, and if the ſide O D, of the triangle O D F, be equal to A B, the tranſverſe axis, and O D F be an obtuſe angle, then the ſtraight line O D produced will meet the curve of the hyperbola B P, in which the focus F is ſituated. For let O D be produced to K, and make the angle D F L equal to the angle F D K. Then, as by hypotheſis O D F is an obtuſe angle, K D F is an acute angle, and therefore, as the angle D F L is equal to it, the ſtraight lines D K, F L, being produced, will meet. Let them meet in P, and (6. i.) P F will be equal to P D. Conſequently, as the difference of P O, P F is equal to O D, or A B, the point P is in the curve of the hyperbola, by the preceding Corollary.

PROP. XV.

If from any point in the curve of an ellipſe, or hyperbola, two ſtraight lines be drawn to the foci, the ſtraight line biſecting the angle adjacent to that contained by them will touch the ellipſe; but the ſtraight line biſecting the angle contained by them will touch the hyperbola.

Fig. 62. 63. From the point P, in the curve of the ellipſe or hyperbola B P, let two ſtraight lines P F, P O, be drawn to the foci F, O, and in the ellipſe let O P be produced to D; the ſtraight line P E biſecting the angle F P D, ad-

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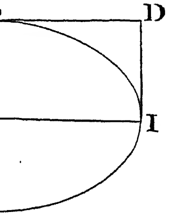
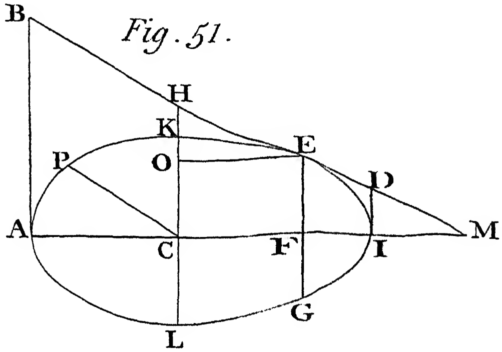


Fig. 51.



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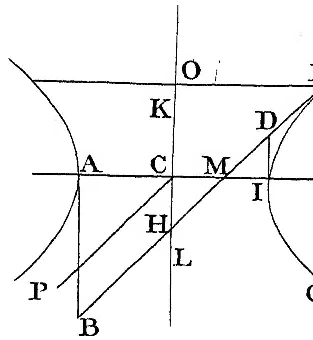

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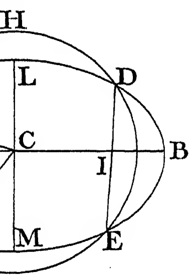


Fig. 58.

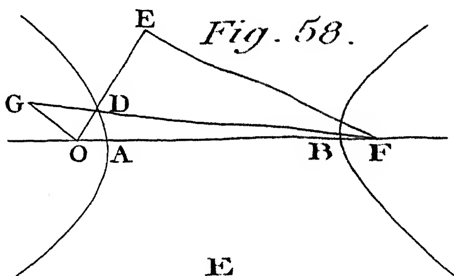


Fig. 56.

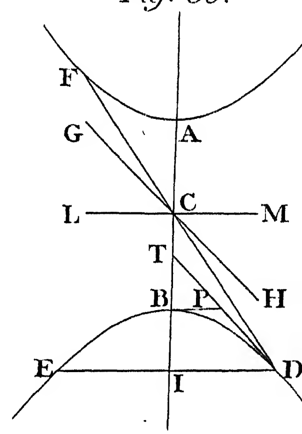


Fig. 54.

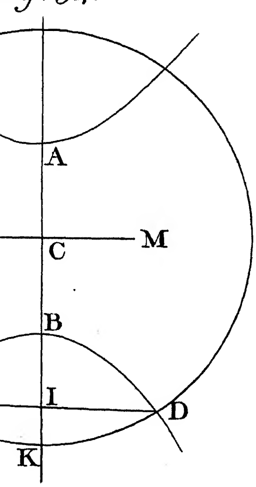


Fig. 57.

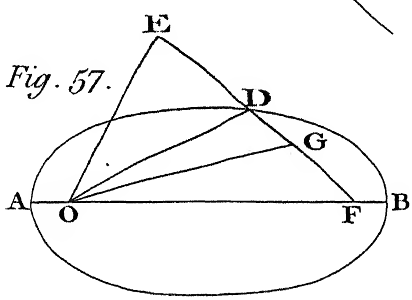


Fig. 55.

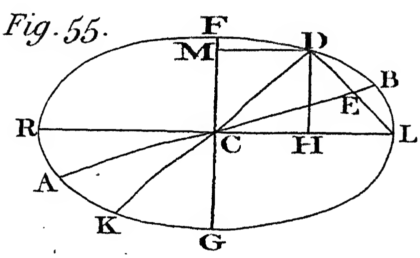
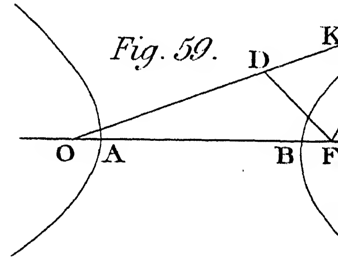


Fig. 59.



adjacent to F P O in the ellipſe, touches the ellipſe; but the ſtraight line P E, biſecting the angle F P O in the hyperbola, will touch the hyperbola. BOOK II.

In the ellipſe let P D, the part of O P produced, be equal to P F. Draw F D, and let it meet P E in E. Then, as F P, P D are equal, and P E common to the two triangles P E F, P E D, and the angle F P E equal to the angle D P E, the ſide F E (4. i.) is equal to D E, and the angles F E P, D E P are equal. In E P take any point G, and draw G O, G F, G D. Then, as F E is equal to E D, and as the angles F E G, D E G are equal, we have F G (4. i.) equal to D G. But (20. i.) D G, G O together are greater than D O, or O P, P F together; and therefore, by Prop. XIII. D G, G O together are greater than A B the tranſverſe axis. Conſequently G O, G F together are greater than A B, and therefore, by Cor. 1. to Prop. XIV. the point G is without the ellipſe B P, and conſequently P E touches it in P. Fig. 62.

In the hyperbola take P D in P O equal to P F. Draw F D, and let it meet P E in E. Then, as P F, P D are equal, and as P E is common to the two triangles F P E, D P E, and as the angles F P E, D P E are equal, the ſide F E (4. i.) is equal to the ſide E D, and the angles F E P, D E P are alſo equal. In E P take any point G, and draw G O, G D, G F. Then, as F E, E D are equal, and as the angles F E G, D E G are alſo equal, and E G common to the two triangles F E G, D E G, the ſide F G (4. i.) is equal to the ſide D G. Alſo, as P D is equal to P F, by Prop. XIII. D O is equal to A B, the tranſverſe axis. But G D, D O together (20. i.) are greater than G O, and therefore G F and A B together are greater than G O. Conſequently the difference between G O and G F is leſs than A B, and therefore, by Cor. 1. to Prop. XIV. the point G is without the hyperbola, and P E touches the hyperbola. Fig. 63.

Cor.

BOOK II.

Cor. 1. From this Prop. and Prop. VI. Book I. it is evident, that if a ſtraight line touch an ellipſe or hyperbola, and ſtraight lines be drawn from the point of contact to the foci, in the ellipſe the tangent will biſect the angle adjacent to that contained by theſe two ſtraight lines drawn to the foci; but in the hyperbola the tangent will biſect the angle contained by theſe two ſtraight lines drawn to the foci. In the ellipſe the angle O P G (15. i.) is equal to the angle F P E.

Fig. 62.

Fig. 62. 63. *Cor.* 2. If from the foci O, F of an ellipſe, or hyperbola, two ſtraight lines, O D, F D be drawn to a third point D, of which O D, one of them, is equal to the tranſverſe axis A B, and if the other F D be biſected in E, by a ſtraight line P E at right angles; the perpendicular P E will ſomewhere touch the ſection, provided, in the hyperbola, O D F be an obtuſe angle. And, on the contrary, if P E touch the ſection and biſect F D in E at right angles, then O D will be equal to the tranſverſe axis. This is evident from Cor. 2. Prop. XIV. Prop. VI. Book I. and the above demonſtration.

Fig. 64. 65. *Cor.* 3. The reſt remaining as above, let F G be at right angles to the ſtraight line L G, touching the ellipſe or hyperbola in L, and let F G be produced to H, ſo that G H may be equal to F G; then a ſtraight line, biſecting D H at right angles, will paſs through the focus O. For, by the preceding Cor. O H is equal to the tranſverſe axis, and conſequently equal to O D. If therefore O K be drawn, biſecting D H in K, the angles (8. i.) O K D, O K H will be equal. Hence the Cor. is evident.

Fig. 66. 67. *Cor.* 4. The reſt remaining as in the demonſtration of the Propoſition, let D F be ſo divided in L, that D L may be to L F as A B to O F, or, which is the ſame thing, as D O to O F, and in D F, produced as in the figures, let D N be to N F as A B to O F, and then a circle deſcribed upon

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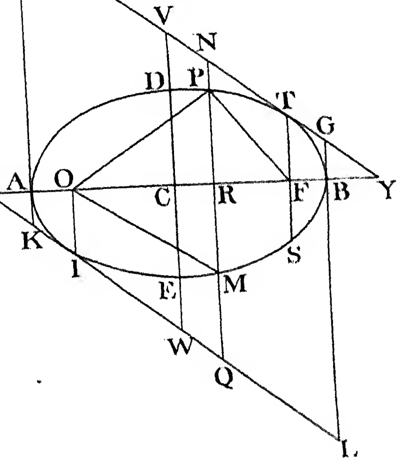

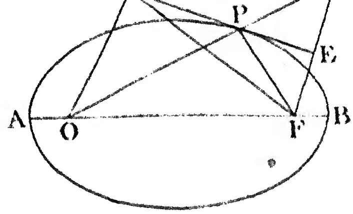

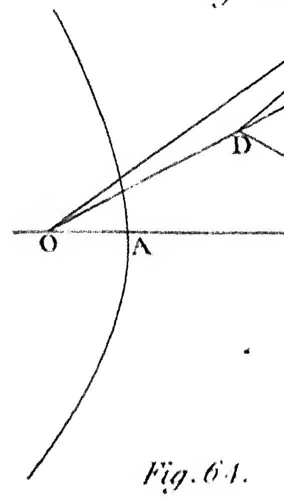

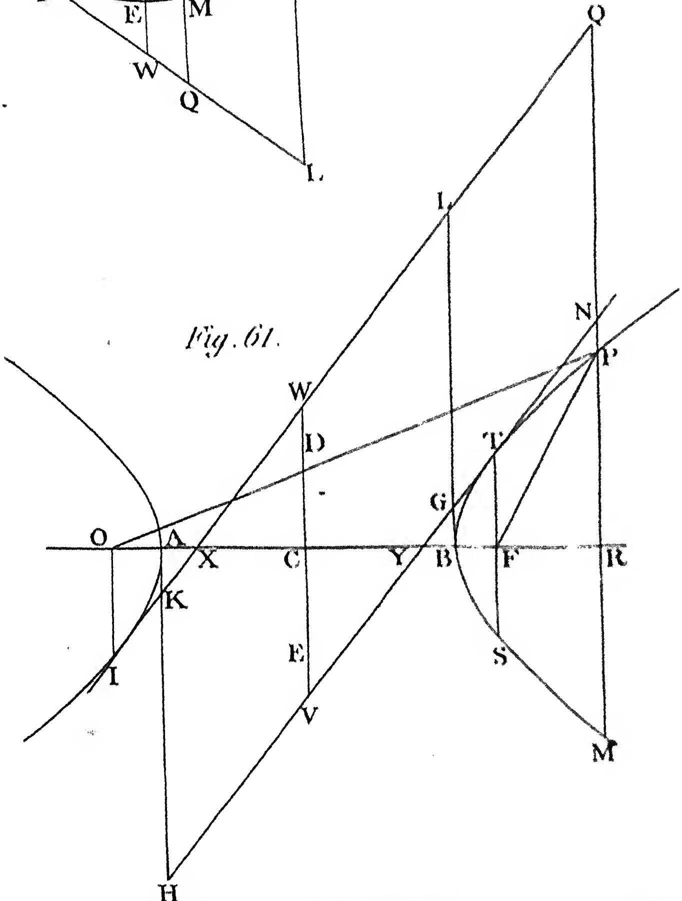

Fig. 61.

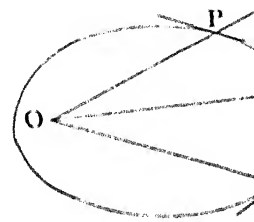

Fig. 64.

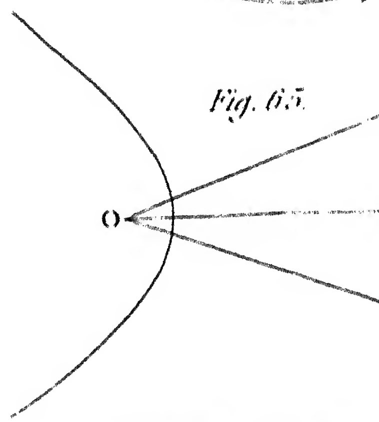

Fig. 65.

L N as a diameter will paſs through the focus O. For in the ellipſe produce D O to M, but in the hyperbola produce F O to M, and draw O L, O N. Then the angles D O F, F O M* (3. vi.) in the ellipſe are biſected by the ſtraight lines O L, O N; and as the angles D O F, F O M together (13. i.) are equal to two right angles, the angles L O F, F O N together are equal to a right angle. But in the hyperbola the angles D O F, D O M* are biſected (3. vi.) by the ſtraight lines O L, O N; and as the angles D O F, D O M together are equal to two right angles, the angles L O D, D O N together are equal to a right angle. In either caſe therefore the angle L O N is a right angle, and conſequently (31. iii. and 21. i.) a circle deſcribed about L N as a diameter will paſs through O.

Sir Iſaac Newton makes much uſe of the properties expreſſed in the three laſt Corollaries. See the Principia, Sect. IV. Book I.

PROP. XVI.

If a ſtraight line touching an ellipſe or hyperbola meet a ſtraight line drawn from either of the foci, and be at right angles to it, the ſtraight line joining the center and the point of concourſe will be equal to the ſemitranſverſe axis: or, if a ſtraight line touch an ellipſe or hyperbola, and ſtraight lines be drawn from the point of contact to the foci, a ſtraight line drawn from the center to the tangent, and parallel to either of the two drawn to the foci, will be equal to the ſemitranſverſe axis.

Part I. Let the ſtraight line G P, touching the ellipſe or hyperbola P B in the point P, meet in the points G, Fig. 68. 69.

* That the angle F O M in the ellipſe, and the angle D O M in the hyperbola, is biſected by O N, is proved by Simſon, in his Prop. A. in the ſixth Book of his edition of Euclid.

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draw C G, C E to the points of concourſe; each of the ſtraight lines C G, C E is equal to A C or C B, the ſemi-tranſverſe axis.

For, draw F P, O P, and let F P, or F P produced, meet O E produced in D. Then, as P E O is a right angle, P E D is a right angle; and as P E is common to the two triangles P E O, P E D, and as, by Cor. I. to Prop. XV. the angles O P E, D P E are equal, the ſide P D (26. i.) is equal to the ſide P O, and D E is equal to E O. Conſequently, by Prop. XIII. F D is equal to the tranſverſe axis A B; and as F C, C O are equal, and D E equal to E O, F O : C O :: D O : E O. The ſtraight lines F D, (2. vi.) C E are therefore parallel, and O F : F D :: O C : C E. But O C is the half of O F, and therefore C E is equal to the half of F D. Conſequently C E is equal to A C or C B, the ſemitranſverſe axis. If F G, P O be produced till they meet in H, it may be proved, in the ſame manner, that F G is equal to G H, F P to P H, H O to A B, and C G to A C or C B.

Fig. 68. 69. Part II. Let the ſtraight line G P touch the ellipſe or hyperbola P B in the point P, and let P F, P O be ſtraight lines drawn from the point of contact to the foci F, O. Let C be the center, and draw C E parallel to F P, and C G parallel to O P, and let C E meet the tangent in E, and C G meet it in G; each of the ſtraight lines C E, C G is equal to A C or C B, the ſemi-tranſverſe axis.

For draw O E, and, being produced, let it meet F P, or F P produced, in D. Then, as F D, C E are parallel, (2. vi.) F C : C O :: D E : E O, and as, by Cor. to Def. XI. F C, C O are equal, D E is equal to E O. By Cor. I. to Prop. XV. the angles O P E, D P E are equal, and therefore (3. vi.) O E : E D :: O P : P D, and as

O E,

O E, E D are equal, O P is equal to P D. Confequently, by Prop. XIII. F D is equal to A B, the tranfverfe axis; and as F D, C E are parallel, F O : F D :: C O : C E. But C O is the half of F O, and therefore C E is the half of F D. Confequently C E is equal to A C or C B, the femitranfverfe axis. If F G be drawn, and, being produced, meet O P produced in H, it may be proved, in the fame manner, that F G is equal to G H, F P to P H, and C G to A C or C B.

Cor. The reft remaining as above, if the ftraight line I K, drawn through C the center, and parallel to the tangent G P, meet P F in I and P O in K, the fegments P I, P K are equal; and each of them is equal to A C or C B. For (34. i.) P K is equal to C G and P I is equal to C E.

The demonftrations of the 11th and 12th Propofitions of the firft Book of the Principia depend, in a very confiderable degree, on this property.

PROP. XVII.

The rectangle contained under two ftraight lines, drawn from the foci of an ellipfe or hyperbola to a tangent, and at right angles to it, is equal to the fquare of the femiconjugate axis. And the rectangle contained under two ftraight lines, drawn from the tranfverfe axis of an ellipfe or hyperbola to a tangent, and at right angles to it, is equal to the fquare of the femiconjugate axis, if one of them be drawn from the center, and the other meet the tangent in the point of contact.

Part I. Let the ftraight lines F G, O E, drawn from the foci F, O of the ellipfe or hyperbola P B, meet in the points G, E, the ftraight line G E which touches the fection in P, and let them be at right angles to the tangent G E, and C being the center, let C D be the femiconjugate Fig. 70. 71.

BOOK II. miconjugate axis; the rectangle under F G, O E is equal to the ſquare of C D.

Draw E C, and, being produced, let it meet F G, or F G produced, in H. Then as F G, O E are at right angles to G P E, they are parallel to one another, (28. i.) and (29. i.) the angles C F H, C O E are equal. The triangles (15. and 32. i.) C F H, C O E are therefore equiangular, and, Cor. 1. to Def. XI. C F is equal to C O. Conſequently (26. i.) C H is equal to C E, and F H is equal to O E. If therefore, with C as a center, and C A or C B as a diſtance, a circle be deſcribed, it will paſs through the points E, G, by Prop. XVI. and conſequently through H; and therefore (35. and 36. iii.) the rectangle under G F, F H is equal to the rectangle under A F, F B. But as F H is equal to O E, the rectangle under G F, E O is equal to the rectangle under A F, F B; and as, by the eleventh Definition, the rectangle under A F, F B is equal to the ſquare of C D, the rectangle under G F, E O is alſo equal to the ſquare of C D.

Part II. Let A B be the tranſverſe axis of the ellipſe or hyperbola P B, of which C is the center, and let G E touch the ſection in the point P; the rectangle under the ſtraight lines C K, M P, drawn from the tranſverſe axis A B to the tangent G E, and at right angles to it, is equal to the ſquare of C D, the ſemiconjugate axis.

Let the conjugate axis meet the tangent in the point I; and from the point P draw P N an ordinate to A B, and P L an ordinate to D C I. Then, as C K, M P are at right angles to the tangent G E, they are (28. i.) parallel to one another, and, by Cor. 2. Prop. IV. as P N is an ordinate to A B it is parallel to D C I, and as P L is an ordinate to D C I it is parallel to A B. Conſequently (29. i.) the angle K C B is equal to the angle P M N; and as I C B, P N M are right angles, the angles I C K,

K C B

K C B together are equal to the angles M P N, P M N together, and therefore the angles I C K, M P N are equal, and the triangles I C K, M P N are equiangular. Hence C K : C I : : P N : P M. But (34. i.) P N is equal to C L, and therefore C K : C I : : C L : P M, and (16. vi.) C K × P M is equal to C I × C L. Consequently, as the rectangle under C I, C L, by Prop. VII. (and 17. vi.) is equal to the square of C D, the rectangle under C K, P M is also equal to the square of C D.

PROP. XVIII.

If a straight line touching an ellipse or hyperbola be limited by tangents passing through the vertices of the transverse axis, the circumference of a circle described about it as a diameter will pass through the foci; and the rectangle under the two straight lines drawn from the point in which it touches the section to the foci will be equal to the square of the semidiameter parallel to it.

Part I. Let the straight line G E touch the ellipse or hyperbola P B in the point P, and meet in the points G, E the tangents A G, B E, passing through the vertices A, B of the transverse axis A B; the circumference of a circle described about G E as a diameter will pass through the foci F, O. Fig. 72. 73.

For, by the seventh Definition, and Cor. 2. Prop. IV. G A B, E B A are right angles; and, by the eleventh Definition, and Prop. IX. the rectangle under A G, B E is equal to the rectangle under A O, O B. Consequently (17. vi.) E B : B O : : A O : A G, and therefore the straight lines E O, G O being drawn, the angle E O B (6. vi.) is equal to the angle A G O, and the angle B E O is equal to the angle A O G. The angles E O B, A O G together are therefore equal to a right angle, and consequently (32. i.) the angle G O E is a

 right

BOOK II. right angle. If therefore a circle be defcribed about G E as a diameter, it is evident (from Prop. 31. iii. and 21. i.) that its circumference muft pafs through the focus O; and in the fame way it may be proved, that it muft pafs through the focus F.

Part II. The reft remaining as above, let P O, P F be drawn to the foci O, F, and let C D be the femidiameter parallel to G E; the rectangle under P F, P O is equal to the fquare of C D.

For, draw O I perpendicular to G E, and, being produced, let it meet P F, or P F produced, in H. Then, by Cor. 1. to Prop. XV. the angles O P I, H P I are equal, and the angle O I P is equal to the angle H I P, each of them being a right angle, and P I is common to the two triangles O P I, H P I. Confequently (26. i.) P O is equal to P H, and O I to I H; and it is evident (from 3. iii.) that the point H muft be in the circumference of the circle defcribed about G E as a diameter, and paffing through F, O, according to Part I. The rectangle under F P, P H, or that under F P, P O, (35. and 36. iii.) is therefore equal to the rectangle under G P, P E. Confequently, by Prop. IX. the rectangle under F P, P O is equal to the fquare of C D.

PROP. XIX.

A ftraight line drawn from either of the foci of an ellipfe or hyperbola, perpendicular to a tangent, is to a ftraight line drawn from the fame focus to the point of contact, as the femiconjugate axis to the femidiameter parallel to the tangent.

Fig. 70. 71. Let P B be an ellipfe or hyperbola, of which the foci are F, O, and let G E touch the fection in P, and let F G be perpendicular to it; the perpendicular F G is to the ftraight line F P, joining the focus F and point of con-

contact, as C D the semiconjugate axis to C R the semidiameter parallel to G E.

For, draw O E perpendicular to G E, and draw P O. Then, by Cor. 1. to Prop. XV. the angles F P G, O P E are equal; and F G P, O E P being right angles, the triangles F P G, O P E, are equiangular. Consequently (4. vi.) F G : F P : : O E : O P, and by alternation F G : O E : : F P : O P; and therefore (22. vi.) F G × O E : F P × O P : : FG^2 : FP^2. But, by Prop. XVII. F G × O E is equal to CD^2, and F P × O P is equal to CR^2, by Prop. XVIII. Consequently CD^2 : CR^2 : : FG^2 : FP^2, and therefore (22. vi.) C D : C R : : F G : F P.

Cor. 1. The rest remaining as above, let C E parallel to F P meet the tangent G E in E, and let C K be perpendicular to G E and meet it in K. Then (4. vi.) F G : F P : : C K : C E. But, by Prop. XVI. C E is equal to C B, the semitransverse axis, and therefore, by the above, (and 11. v.) C K : C B : : C D : C R.

Cor. 2. The rest remaining as above, let the straight line P M, perpendicular to the tangent G E, meet the transverse axis A B in M, and then C B will be to C D as C R to P M. For, by the preceding Cor. C K : C B : : C D : C R, and therefore (1. vi.) C K × P M : C B × P M : : CD^2 : C R × C D. But, by Prop. XVII. C K × P M is equal to CD^2, and therefore (14. v.) C B × P M is equal to C R × C D. Consequently (16. vi.) C B : C D : : C R : P M.

PROP. XX.

If a straight line touch an ellipse or hyperbola, and from the point of contact two straight lines be drawn to an axis, the one an ordinate to it, the other perpendicular to the tangent, the segment of the axis between the center and ordinate will be to the segment between the perpendicular and ordinate as the axis to its parameter.

BOOK II.

Fig. 74. 75.

Let the ſtraight line I R touch the ellip bola P B in the point P; and A B being th and D E the conjugate axis, and C the ce be an ordinate to A B, and P G an ordinate let the ſtraight line M P be perpendicular meet A B in K and D E in M; then C H A B to its parameter, and C G is to G M a parameter.

Let the tangent I R meet A B in R, an Then, by Cor. 2. Prop. VII. the rectangle H R is equal to the rectangle under A I; R P K being a right angle, and P H being gles to K R, the rectangle under K H, H R the ſquare of P H (Cor. 8. vi. and 16. vi.). C H : K H : : C H × H R : K H × H R; an on account of the equals, (and 11. v.) C A H × H B : P H². Conſequently, as by A H × H B is to P H² as A B to its paramete K H (11. v.) as A B to its parameter.

Again (1. vi.) in the ellipſe, C G : G M : G M × G I; and as above, by Cor. 2. Pro rectangle under C G, G I is equal to the re der E G, G D, and (Cor. 8. vi.) the recta G M, G I is equal to the ſquare of P G. C C G : G M : : E G × G D : P G²; and th Prop. VI. (and 11. v.) C G is to G M as D parameter.

Laſtly, in the hyperbola, as, by Cor. 2 P G is parallel to A B, and P H parallel to D G M as K P to P M, and therefore as K H to (1. vi.) K H : H C : : K H × R H : H C × (Cor. 8. vi. and 17. vi.) K H × R H is equa and, by Cor. 2. Prop. VII. H C × R H is eq × H B. Conſequently (11. v.) C G : G M A H × H B. But, by Def. VIII. P H² is to

66.

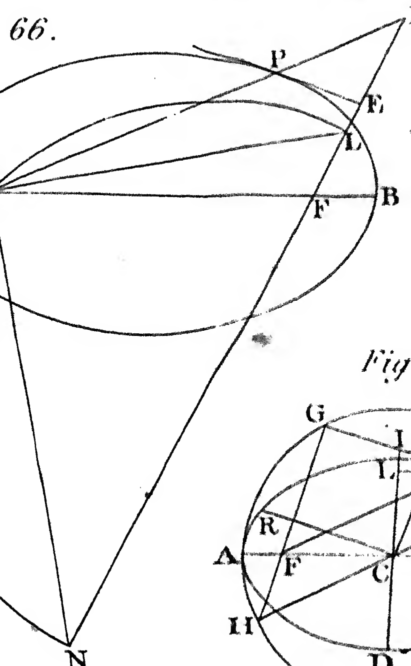


Fig. 68.

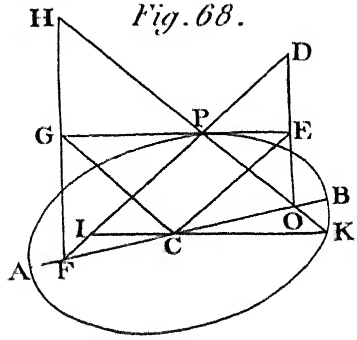


Fig. 70.

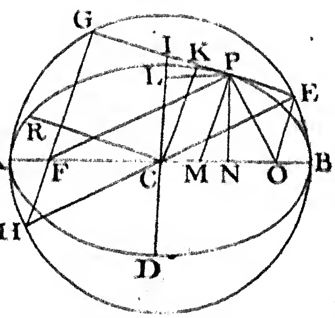


Fig. 69.

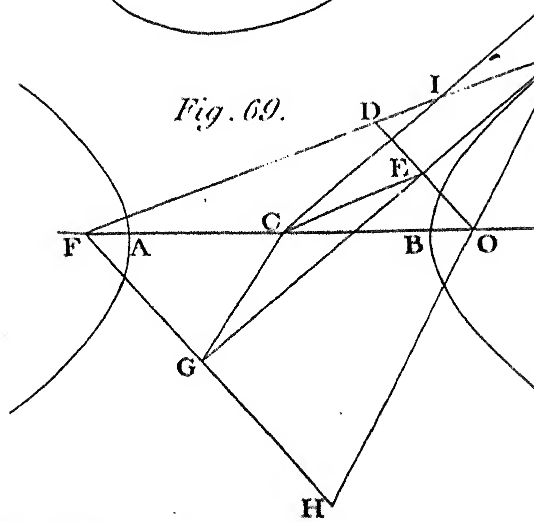


Fig. 67.

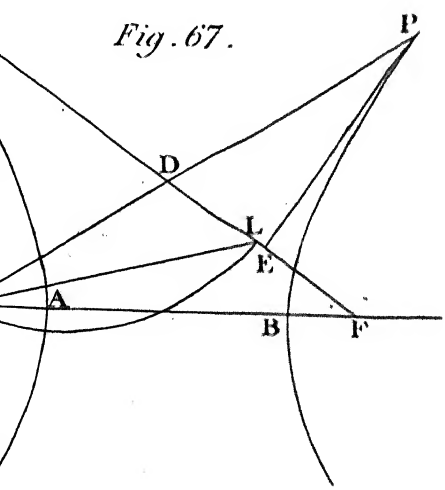
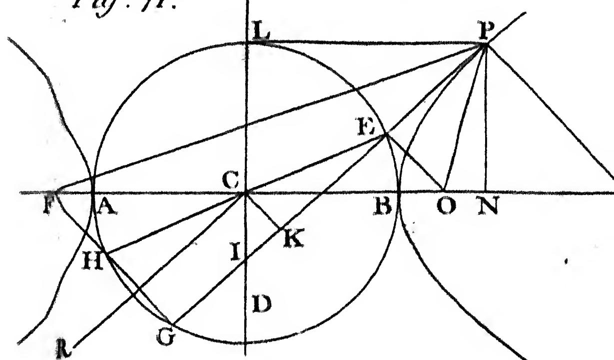


Fig. 71.



is evident from Def. IX. D E is to A B as D E is to its parameter. Consequently C G is to G M as D E is to its parameter.

PROP. XXI.

If a straight line touch an ellipse or hyperbola, and a straight line be drawn from the point of contact at right angles to it, and meet the axes, the rectangle under the segments of the perpendicular, between the point of contact and the axes, will be equal to the square of the semidiameter parallel to the tangent.

Let the straight line I R touch the ellipse or hyperbola P B in the point P, and let the straight line M P, at right angles to I R, meet the transverse axis A B in K, and the conjugate axis D E in M, and, C being the center, let C L be the semidiameter parallel to I R; the rectangle under P M, P K is equal to the square of C L. Fig. 74. 75.

For, by Cor. 2. Prop. XIX. and inversion, P K : C L :: C D : C B, and therefore $PK^2 : CL^2 :: CD^2 : CB^2$. And P G being drawn an ordinate to D E, by Prop. XX. C G is to G M as D E to its parameter. But, by Def. IX. (and Cor. 2. to 20. vi.) D E is to its parameter as DE^2 to AB^2, or (15. v.) as CD^2 to CB^2; and as P G is parallel to A B, (2. vi.) C G : G M :: P K : P M. Consequently (11. v.) $PK : PM :: CD^2 : CB^2$; and therefore (1. vi.) $PK^2 : PM \times PK :: CD^2 : CB^2$. But, by the above, $PK^2 : CL^2 :: CD^2 : CB^2$; and therefore (11. v.) $PK^2 : PM \times PK :: PK^2 : CL^2$. Consequently (14. v.) $PM \times PK$ is equal to CL^2.

Cor. By the above, and Prop. XVIII. the rectangle under P M, P K is equal to the rectangle under P O, P F, the straight lines drawn from P the point of contact to the foci O, F.

BOOK II.

PROP. XXII.

If a circle be deſcribed about the tranſverſe axis of an ellipſe as a diameter, a polygon may be inſcribed in it, and a correſponding polygon in the ellipſe, ſo that the polygon in the circle ſhall be to that in the ellipſe as the tranſverſe axis to the conjugate axis.

Fig. 76. Let A B be the tranſverſe, and F G the conjugate axis of the ellipſe A F B G, and about A B as a diameter let the circle A D B E be deſcribed; a polygon may be inſcribed in A D B E, and a correſponding one in the ellipſe A F B G, ſo that the polygon in the circle ſhall be to that in the ellipſe as A B to F G.

For, let C be the common center of the ellipſe and circle, and produce F G till it meet the circumference of the circle in D, E. Let K, M be points in the circumference, and draw D K, K M, M A. Draw K P, M O parallel to D C, and let them meet A C in P, O, and the curve of the ellipſe in L, N, and draw F L, L N, N A. Draw K H, L I parallel to A B, and let them meet D C in H, I. Then P I, P H are parallelograms, and (34. i.) C H is equal to P K, and C I equal to P L; and K P, M O are perpendicular to A C, and L P, N O are ordinates to A B, by Cor. 2. Prop. IV. By Prop. V. $AC^2 : CF^2 :: AP \times PB : PL^2$, and therefore (35. iii.) as $AP \times PB$ is equal to KP^2, $AC^2 : CF^2 :: KP^2 : PL^2$, and (22. vi.) A C or D C : C F :: K P or C H : P L or C I*. Conſequently (19. v.) D C : F C :: D H : F I; and D H : F I :: C H : C I. But (1. vi.) C H : C I :: parallelogram P H : parallelogram P I; and D H : F I :: the triangle D K H : the triangle F L I. Conſequently

* This is the property referred to by writers on the Orthographical Projection of the Sphere, when they prove, that a circle, not parallel to the plane of projection, is projected into an ellipſe.

(II.

(11. and 12. v.) D C : F C : : the trapezium D K P C : the trapezium F L P C. In the same manner it may be demonstrated that D C : F C : : the trapezium K M O P : the trapezium L N O P; and also that D C : F C : : the triangle M A O : the triangle N A O. But (15. v.) D C : F C : : A B : F G, and therefore (12. v.) A B : F G : : the polygon D K M A C : the polygon F L N A C. Consequently, as inscriptions may be made in a similar manner all round the circle and ellipse, the Prop. is evident. BOOK II.

Cor. A polygon may be inscribed in an ellipse, which shall be deficient from the ellipse by a superficies less than any given superficies. For, the rest remaining as above, if a straight line parallel to L F be drawn to touch the ellipse, and meet C F and P L produced, and from F, L straight lines be drawn to the point of contact, the triangle thus formed will be equal to half the parallelogram contained by L F, the tangent parallel to it, and C F, P L produced. This triangle therefore will be greater than half the elliptic segment contained under the curve L F, and the straight line L F. Such an inscription therefore being made all round the ellipse, the Cor is evident (from 1. x.).

PROP. XXIII.

If the transverse axis of an ellipse be also a diameter of a circle, the ellipse will be to the circle as the conjugate axis to the transverse axis.

Let A B be the transverse axis of the ellipse A F B G, and also a diameter of the circle A D B E; the ellipse is to the circle as F G the conjugate axis to A B the transverse axis.

For, every thing remaining as in Prop. XXII. let the circle Q R S be to the circle A D B E as F G to A B; and then, if it can be proved that the circle Q R S is Fig. 76.

equal

BOOK II.

equal to the ellipse A F B G, the truth of the Proposition will be manifest. If the circle Q R S be not equal to the ellipse, let it first, if possible, be greater. Then it is possible to inscribe in the circle Q R S a polygon, having an even number of sides, and greater than the ellipse A F B G. Let it be understood to be inscribed, and let a polygon similar to it be supposed to be inscribed in the circle A D B E; and from the angular points of the polygon in A D B E let straight lines be drawn parallel to D E. Let the points in which these parallel lines cut the curve of the ellipse be joined, and then a polygon will be inscribed in the ellipse corresponding to the polygon in the circle A D B E, as in the last Proposition; and A B : F G : : the polygon inscribed in the circle A D B E : the polygon inscribed in the ellipse. But, by hypothesis and inversion, the circle A D B E : the circle Q R S : : A B : F G; and therefore (11. v.) the polygon inscribed in the circle A D B E : the polygon inscribed in the ellipse : : the circle A D B E : the circle Q R S. Consequently (1. and 2. xii.) the polygon inscribed in the circle A D B E : the polygon inscribed in the ellipse : : the polygon inscribed in the circle A D B E : the polygon inscribed in the circle Q R S. The polygon (14. v.) inscribed in the ellipse is therefore equal to the polygon inscribed in the circle Q R S: which is absurd; for, by the present hypothesis, the polygon inscribed in the circle Q R S is greater than the ellipse.

Secondly, if it be possible, let the circle Q R S be less than the ellipse. Then it is possible, by Cor. Prop. XXII. to inscribe in the ellipse a polygon greater than the circle Q R S, and a polygon corresponding to it in the circle A D B E; and to inscribe in the circle Q R S a polygon similar to the polygon inscribed in the circle A D B E. Let such polygons be supposed to be so inscribed.

lygon infcribed in the circle Q R S, contrary to the conftruction which has now been fuppofed to be made. The circle Q R S therefore is equal to the ellipfe A F B G, and therefore the ellipfe A F B G is to the circle A D B E as F G to A B.

Cor. 1. An ellipfe is equal to a circle, whofe diameter is a mean proportional between its axes. For, by the above, F G : A B : : the ellipfe A F B G, or the circle Q R S : the circle A D B E. But (1. vi.) F G : A B : : F G × A B : A B^2; and (2. xii.) the circle Q R S : the circle A D B E : : Q S^2 : A B^2, Q S being the diameter of the circle Q R S. Confequently (11. v.) the ellipfe A F B G : circle A D B E : : Q S^2 : A B^2 : : F G × A B : A B^2 : and (14. v.) Q S^2 is equal to F G × A B.

Cor. 2. From the preceding (and 17. vi. and 2. xii.) it is evident, that the areas of two ellipfes are to one another as the rectangles under their axes.

The Cor. to Prop. XIV. Lib. I. of the Principia depends, in a great degree, upon this truth.

PROP. XXIV.

If from a point in the conjugate axis of an ellipfe a ftraight line, equal to the difference of the femiaxes, be drawn to a point in the tranfverfe axis, and be produced beyond the tranfverfe axis, fo that the part produced be equal to the femiconjugate axis, the extremity of the part produced will be in the curve of the fection. Or, if from a point in the conjugate axis of an ellipfe a ftraight line, equal to the fum of the femiaxes, be drawn to a point in the tranfverfe axis, and if this line be fo cut that the fegment between the tranfverfe axis and the point

BOOK II.

point of section be equal to the semiconjugate axis, the point of section will be in the curve.

Fig. 77. 78.

Let A D B E be an ellipse, of which A B is the transverse, and D E is the conjugate axis, and C is the center. Let F be a point in the conjugate, and G be a point in the transverse axis, and let the straight line F G be equal to the difference or sum of C B, C D. When F G is equal to the difference of C B, C D, as in Fig. 78. let F G be produced beyond A B to H, so that G H be equal to C D; but when F G is equal to the sum of C B, C D, as in Fig. 77. let the segment G H be equal to C D; and in either case the point H will be in the curve of the ellipse.

For through the center C draw the straight line C K parallel to F H. Through H draw the straight line H K parallel to D E, and let it meet C K in K, and A B in I. Then (34. i.) the straight line C K is equal to F H. But as F G is equal to the sum or difference of C B, C D, and as G H is equal to C D, the straight line F H is equal to C B; and consequently C K is equal to C B. With C, therefore, as a center, and C B as a distance, let a circle be described, and it will pass through K. Again, on account of the similar triangles C K I, G H I, $CK^2 : GH^2 :: KI^2 : HI^2$; and therefore, on account of the equals, $CB^2 : CD^2 :: KI^2 : HI^2$. But (3. and 35. iii.) the square of K I is equal to the rectangle under A I, I B; and therefore $CB^2 : CD^2 :: AI \times IB : HI^2$. The point H is therefore in the curve, by Cor. 1. Prop. V. (and 9. v.) for H I is parallel to the ordinates of A B.

SCHOLIUM.

The instrument called by some *the trammels*, and by others

others the *elliptic compaſſes*, uſed by cabinet-makers, &c. for deſcribing the curves of ellipſes, are conſtructed on the property demonſtrated in this Propoſition. As the trammels are in general uſe, it is needleſs to give a deſcription of them in this place. Lathes for making picture-frames, and ornaments of an elliptical form, are conſtructed on the ſame property. BOOK II.

A GEO-

A
GEOMETRICAL TREATISE
OF
CONIC SECTIONS.

BOOK III.

Of the Parabola, the Directrices of the Sections, the Asymptotes of the Hyperbola, Conjugate Hyperbolas, and of hyperbolic Sectors and Trapezia.

DEFINITIONS.

I.

THE section F D C being a parabola, and V B E its vertical plane, as in the 15th and 16th Definitions in the first Book, any straight line, as D I, in the parabola parallel to V B, the side in which V B E touches the cone, is called a *Diameter* of the parabola. Fig. 79.

Cor. 1. From this Definition (and 9. xi.) the diameters of a parabola are parallel to one another; and, by Prop. XIV. Book I. any straight line drawn in the plane of a parabola, parallel to a diameter, will meet the curve in one point, and in one point only, and, by this Definition, it will itself be a diameter.

Cor.

BOOK III.

Cor. 2. Any ſtraight line in a parabola, not parallel to a diameter, will meet the curve in two points. For any ſtraight line drawn through V, the vertex of the cone, and in the vertical plane, and not in the ſame direction with V B, will fall without the oppoſite cones; and by the demonſtration of the ſecond part of Prop. VIII. of the firſt Book, one plane may be drawn through this ſtraight line to touch the conical ſuperficies, and cut the plane of the parabola. The interſection alſo of this plane with the plane of the parabola will touch the parabola, and any ſtraight line in the parabola parallel to this tangent will meet the curve in two points, by Cor. 1. Prop. VIII. Book I. Hence (16. xi.) the Cor. is evident.

II.

The point in which a diameter of a parabola meets the curve is called the *Vertex* of the diameter.

III.

If a ſtraight line terminated by the curve of a parabola be biſected by a diameter, it is called a *Double Ordinate* to that diameter; and its half is ſimply called an *Ordinate* to it.

IV.

The ſegment of a diameter between its vertex and an ordinate is called an *Abſciſs* of that diameter.

V.

The diameter of a parabola, which cuts its ordinates at right angles, is called the *Axis* of the parabola.

VI.

A third proportional to an abſciſs of a diameter of a parabola, and the correſponding ordinate, is called the *Parameter*, or *Latus Rectum* of the diameter. The parameter of the axis is frequently called the *Principal Parameter*, or *Latus Rectum*.

PROP.

PROP. I.

BOOK III.

If each of two diameters of a parabola meet a ſtraight line, and if each of theſe ſtraight lines cut, or one of them cut and the other touch the parabola, and if theſe two ſtraight lines be parallel; then the ſegment of the one diameter, between its vertex and the line which it meets, will be to the ſegment of the other between its vertex and the line which it meets, as the ſquare of the line which meets the firſt mentioned diameter if a tangent, or the rectangle under its ſegments if a ſecant, to the ſquare of the line which meets the other diameter if a tangent, or the rectangle under its ſegments if a ſecant.

Suppoſe D I, G K to be two diameters of a parabola, and let D be the vertex of the one, and G the vertex of the other. Let L P, M N be two parallel ſtraight lines, and let D I meet L P in L, and G K meet M N in M, and let L P, M N either both cut, or one of them cut and the other touch the parabola; then D L is to G M as the ſquare of L P, if a tangent, or the rectangle under its ſegments if a ſecant, to the ſquare of M N, if a tangent, or the rectangle under its ſegments if a ſecant. Fig. 80.

For let F G D C be the parabola as formed in the cone, and D I, G K the diameters mentioned above. Let the parabola cut the plane of the baſe in the ſtraight line F K I C, and let the vertical plane cut it in B E, V B being the ſide along which the vertical plane touches the cone. Then D I, G K are parallel to V B, by the firſt Definition. Through the parallels V B, D I let a plane paſs, and let it cut the cone in the ſide V D A, and the baſe in B I A. Through the parallels V B, G K let a plane paſs, and let it cut the cone in the ſide V G H, and the baſe in B K H. Then (4. vi. and 16. v.) Fig. 79.

D I : V B : : A I : A B, and

V B : G K : : H B : H K.

'H Hence

BOOK III.

Hence (1. vi.) DI : VB :: AI × IB : AB × IB
and VB : GK :: HB × KB : HK × KB.
But (35. iii.) AI × IB is equal to FI × IC, and HK × KB is equal to FK × KC; and, by the feventh Lemma, AB × IB is equal to HB × KB. Confequently, by the above and fubftitution, we have the two following ranks of magnitudes proportionals, taken two and two in the fame order,

DI : VB : GK
FI × IC : AB × IB : FK × KC; and therefore (22. v.) DI : GK :: FI × IC : FK × KC.
Let the ftraight line FKIC have the fame fituation in the parabola in Fig. 80. as in Fig. 79. and firft fuppofe LP, MN to be parallel to the bafe of the cone, or to FC. Then, by Prop. XVI. Book I. DL is to DI as the fquare of LP, if a tangent, or the rectangle under its fegments, if a fecant, to FI × IC; and, by the above, DI is to GK as FI × IC to FK × KC; and again, by Prop. XVI. Book I. GK is to GM as FK × KC to the fquare of MN if a tangent, or the rectangle under its fegments if a fecant. We have therefore

$$DL : DI : GK : GM$$

$$\left.\begin{matrix}t.\ LP^2\\ \text{or}\\ f.\ LP^r\end{matrix}\right\} : FI \times IC : FK \times KC : \left\{\begin{matrix}t.\ MN^2\\ \text{or}\\ f.\ MN^r\end{matrix}\right.$$

Confequently (22. v.) DL is to GM as the fquare of LP if a tangent, or the rectangle under its fegments if a fecant, to the fquare of MN if a tangent, or the rectangle under its fegments if a fecant. Laftly, let LP, MN not be parallel to the bafe of the cone, but let LS, MR be parallel to the bafe, and let them touch or cut either of the conical fuperficies. Then, by the above, DL is to GM as the fquare of LS if a tangent, or the rectangle under its fegments if a fecant, to the fquare of MR if a tangent, or the rectangle under its fegments if

if a ſecant. But, by Prop. XII. Book I. the ſquare of L S if a tangent, or the rectangle under its ſegments if a ſecant, is to the ſquare of M R if a tangent, or the rectangle under its ſegments if a ſecant, as the ſquare of L P if a tangent, or the rectangle under its ſegments if a ſecant, to the ſquare of M N if a tangent, or the rectangle under its ſegments if a ſecant. Conſequently (11. v.) D L is to G M as the ſquare of L P if a tangent, or the rectangle under its ſegments if a ſecant, to the ſquare of M N if a tangent, or the rectangle under its ſegments if a ſecant.

PROP. II.

A diameter of a conic ſection biſects any ſtraight line it meets in the ſection parallel to a tangent paſſing through its vertex; and ordinates to a diameter, and a tangent paſſing through its vertex, are parallel to one another.

In the ellipſe and hyperbola this has been proved, according to Cor. 3. to Prop. III. Book II. In the parabola A B C let the diameter B G cut the ſtraight line A C in the point G, and let A C be parallel to D E touching the parabola in B, the vertex of the diameter B G; the ſtraight line A C is biſected in G. On the contrary, any ſtraight line in the parabola biſected by B G is parallel to A C, or the tangent D E. Fig. 81.

Part I. Through A, C let A D, C E be drawn parallel to B G, and let A D meet the tangent in D, and C E meet it in E. Then, by Cor. 1. Def. I. A D, C E are diameters, and, by Prop. I. $AD : CE :: DB^2 : BE^2$; and (34. i.) as A D, C E are equal, it follows that D B, B E are equal to one another. Conſequently (34. i.) A G is equal to G C.

Part II. If it be poſſible, let the ſtraight line H R in the parabola A B C be biſected by the diameter B G, and not be parallel to A C or D E.

 Through

BOOK III.

Through R draw R M parallel to the diameter B G, and through H draw H L parallel to A C or D E, and let it meet B G in K, the curve again in L, and R M in M. Let B G meet H R in N. Then as H R is biſected in N, and as K N, M R are parallel, (2. vi.) H N : N R : : H K : K M, and H K is equal to K M. But, by Part I. H K is equal to K L, and therefore K L is equal to K M; which is abſurd. Conſequently no ſtraight line in the parabola, unleſs it be parallel to D E, or to the ordinate A C, can be biſected by the diameter B G. Ordinates to the diameter B G muſt therefore be parallel to one another, and to the tangent D E, paſſing through the vertex.

Cor. 1. From hence, and Prop. III. Book II. it is evident, that if a ſtraight line be an ordinate to a diameter, any ſtraight line in the ſection, or oppoſite ſection, and parallel to it, will be an ordinate to the ſame diameter.

Cor. 2. From this Propoſition it is evident, that if a ſtraight line biſect two parallel lines in a conic ſection, it will be a diameter.

Cor. 3. From the above a method of finding a diameter of a given parabola is evident. For two parallel ſtraight lines being drawn in the parabola, a ſtraight line biſecting them, and any ſtraight line parallel to it, will be a diameter.

Cor. 4. The method of finding the axis of a parabola is alſo evident from the above. For, having found a diameter of the parabola, by the preceding Cor. let a ſtraight line at right angles to it be drawn within the parabola, and limited both ways by the curve. Then the diameter biſecting this ſtraight line will be the axis; for, being parallel to the diameter firſt found, it will (29. i.) biſect the ſtraight line in the ſection at right angles.

PROP.

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Fig. 72.

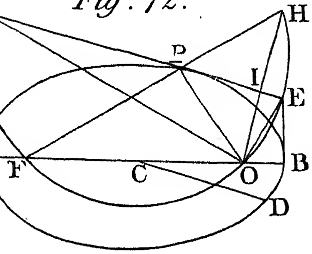
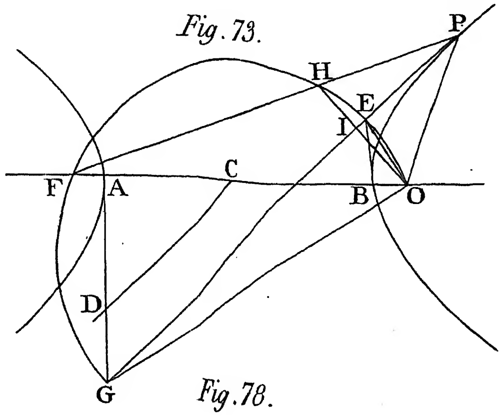


Fig. 73.



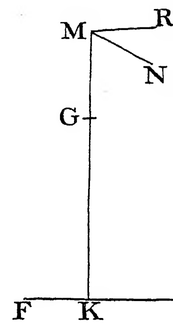


Fig. 77.

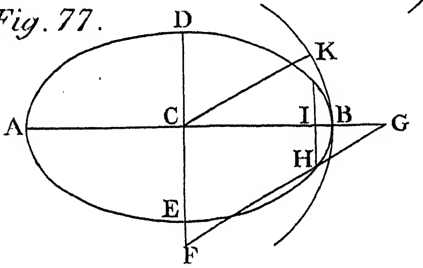
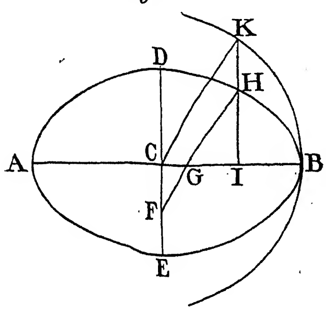


Fig. 78.



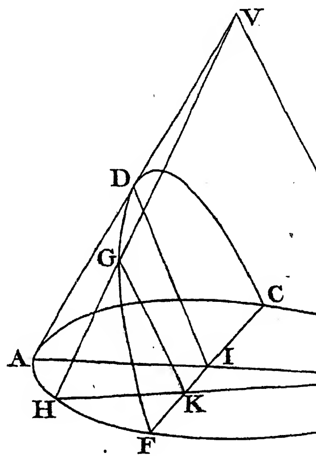


Fig. 74.

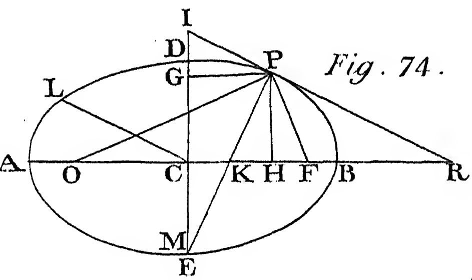


Fig. 75.

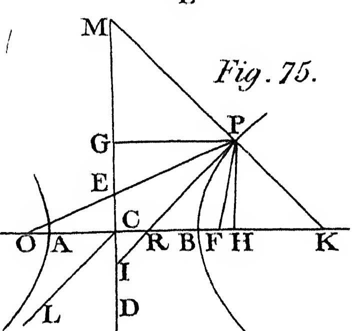
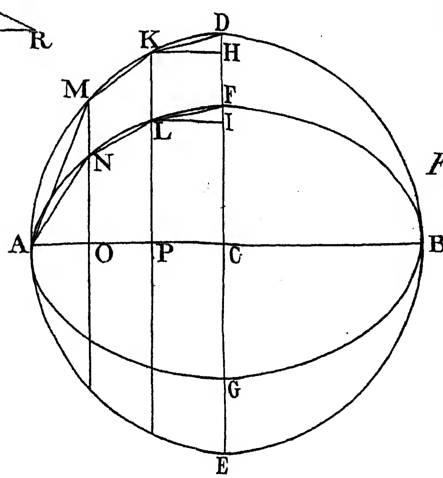
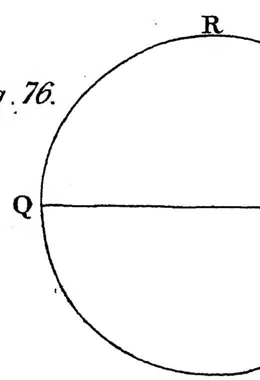


Fig. 76.





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PROP. III.

BOOK III.

The abscisses of a diameter of a parabola are to one another as the squares of the corresponding ordinates; and the square of an ordinate to a diameter of a parabola is equal to the rectangle under the parameter of the diameter, and the abscis corresponding to the ordinate.

Part I. Let B G be a diameter of the parabola A B C, and let A G, H K be ordinates to it, and let them meet it in the points G, K, and let B be the vertex of the diameter; the abscis B G is to the abscis B K as the square of the ordinate A G to the square of the ordinate H K. Fig. 81.

For, as B G is parallel to a side of the cone, in which the section was formed, and as A G, H K, by Prop. II. are parallel, and as they would be bisected in G, K if limited by the curve, this part is evident from Prop. XVI. Book I.*

Part II. The rest remaining as above, let the straight line P be a third proportional to the abscis B G and the corresponding ordinate A G, and consequently the parameter to the diameter B G, according to the sixth Definition; the square of the ordinate H K is equal to the rectangle under P and the abscis B K.

For, by the preceding part, $BG : BK :: AG^2 : HK^2$, and therefore (1. vi.) $P \times BG : P \times BK :: AG^2 : HK^2$. But (17. vi.) $P \times BG$ is equal to AG^2, and consequently (14. v.) $P \times BK$ is equal to HK^2.

Cor. 1. If a straight line touching a parabola meet a diameter, the square of the segment, between the point of contact and the point of concourse, will be equal to

* From this and the second part of Prop. II. writers on projectiles prove, that, if the resistance of the air have no perceptible effect, a projectile must move in the curve of a parabola.

BOOK III. the rectangle under the ſegment of the diameter, between its vertex and the point of concourſe, and the parameter of the diameter, to whoſe ordinates the tangent is parallel. For let D B, touching the parabola in B, meet the diameter A D in D, and let A G parallel to D B meet the diameter B G in G, and let P be the parameter of B G. Then (34. i.) A G is equal to D B, and A D is equal to B G, and, by Prop. II. D B is parallel to the ordinates of B G; and as, by the above, B G × P is equal to A G², B D² is equal to A D × P.

Cor. 2. If a ſtraight line cutting a parabola meet a diameter, the rectangle under its ſegments, between the point of concourſe and the curve, will be equal to the rectangle under the ſegment of the diameter between its vertex and the point of concourſe, and the parameter of the diameter, to whoſe ordinates the ſecant is parallel. For, the reſt remaining as in the preceding Cor. let the tangent B D be parallel to the ſtraight line H M, cutting the parabola in H, L, and meeting the diameter R M in M. Then, by Prop. I. A D : R M : : D B² : H M × M L; and therefore (1. vi.) A D × P : R M × P : : D B² : H M × M L. Conſequently, by the preceding Cor. (and 14. v.) R M × P is equal to H M × M L.

SCHOLIUM.

On account of the equality of the ſquare of H K to the rectangle under P and B K, Apollonius called the ſection a parabola.

From the property demonſtrated above the parabola is frequently denoted by an algebraical equation, in the following manner. Put the parameter of the diameter B K $= p$, the abſciſs B K $= x$, and the ordinate H K $= y$. Then, from the above, $x : y : : y : p$, and $p\,x = y^2$.

PROP.

BOOK III.

PROP. IV.

If each of two ſtraight lines meeting one another touch or cut, or one of them touch and the other cut, a parabola, the ſquare of the firſt of the two if a tangent, or the rectangle under its ſegments if a ſecant, will be to the ſquare of the ſecond if a tangent, or the rectangle under its ſegments if a ſecant, as the parameter of the diameter to whoſe ordinates the firſt is parallel, to the parameter of the diameter to whoſe ordinates the ſecond is parallel.

For, firſt let the ſtraight lines A E, C E, meeting one another in E, touch the parabola in A and C, and let E B be a diameter paſſing through E. Then, by Cor. 1. Prop. III. the ſquare of A E is equal to the rectangle under B E, and the parameter of the diameter to whoſe ordinates A E is parallel; and the ſquare of C E is equal to the rectangle under B E, and the parameter of the diameter to whoſe ordinates C E is parallel. Conſequently (1. vi.) the ſquare of A E is to the ſquare of C E, as the parameter of the diameter to whoſe ordinates A E is parallel to the parameter of the diameter to whoſe ordinates C E is parallel. Fig. 82.

Next, let the ſtraight line G A, touching the parabola in A, meet in G the ſtraight line G K, which cuts the parabola in F, K, and let G D be a diameter paſſing through G. Then, by Cor. 1. Prop. III. the ſquare of A G is equal to the rectangle under D G, and the parameter of the diameter to whoſe ordinates A G is parallel; and, by Cor. 2. Prop. III. the rectangle K G F is equal to the rectangle under D G, and the parameter of the diameter to whoſe ordinates G K is parallel. Conſequently (1. vi.) the ſquare of A G is to the rectangle K G F as the parameter of the diameter to whoſe ordi-

BOOK III. nates A G is parallel, to the parameter of the diameter to whoſe ordinates G K is parallel.

Laſtly, if the ſtraight line G K, cutting the parabola in F, K, meet in the point G the ſtraight line G L, which cuts the parabola in the points H, L, then it may be proved in the ſame way, by means of Cor. 2. Prop. III. that the rectangle K G F is to the rectangle L G H as the parameter of the diameter to whoſe ordinates G K is parallel, to the parameter of the diameter to whoſe ordinates G L is parallel.

PROP. V.

If a ſtraight line touching a parabola meet a diameter, and an ordinate to the diameter paſs through the point of contact, the ſegment of the diameter, between its vertex and the tangent, will be equal to its abſciſs, between its vertex and the ordinate.

Fig. 83. Let the ſtraight line A E, touching the parabola A B C in the point A, meet the diameter B D in the point E, and through the point of contact A let the ordinate A D to B D paſs, and meet B D in D; the ſegment B E between B the vertex and the tangent is equal to the abſciſs B D between the vertex and the ordinate.

For produce A D till it meet the curve in C, and draw C F parallel to E D, and let it meet A E in F. Then, by the third Definition, A C is biſected in D, and, by Cor. 1. to the firſt Definition, C F is a diameter; and, by Prop. I. B E : C F : : A E^2 : A F^2. But C F, D E being parallel, (2. vi.) A D : D C : : A E : E F, and A C being biſected in D, A F is biſected in E, and for the ſame reaſons D E is half of C F. Conſequently A F^2 is equal to four times A E^2 (4. ii.) and, therefore, C F is equal to four times B E; and as C F is double of D E, B E is equal to B D.

Cor.

Cor. If A G, B D be any two diameters of the parabola A B C, and if A D be an ordinate to B D, and B G be an ordinate to A G, the abſciſſes A G, B D will be equal. For let A E touch the parabola in A, and meet the diameter B D in E. Then, by Cor. 1. to the firſt Definition, A G, E B are parallel, and, by Prop. II. A E, G B are parallel. Conſequently (34. i.) A G is equal to E B, and therefore, as by the above E B, B D are equal, A G is equal to B D.

PROP. VI.

If two ſtraight lines touching a conic ſection, or oppoſite hyperbolas, meet one another, the diameter biſecting the line joining the points of contact will paſs through the point of concourſe.

In the ellipſe, hyperbola, or oppoſite hyperbolas, this has been proved in Prop. VIII. Book II. In the parabola A B C let the two ſtraight lines E A, E C touch the ſection in the points A C, and meet one another in E, and let the diameter B D biſect A C, the ſtraight line joining the points of contact in D; the diameter B D will paſs through E. Fig. 83.

For, as A C is biſected by the diameter B D, it is a double ordinate to B D; and therefore, by Prop. V. if B D be produced and meet the tangents, its ſegment between B the vertex and the tangent A E will be equal to its abſciſs B D; and its ſegment between B and the tangent C E will alſo be equal to B D. The diameter B D will therefore meet both the tangents A E, C E in the ſame point, and conſequently will paſs through E, the point of concourſe.

Cor. 1. If two ſtraight lines touching a conic ſection, or oppoſite hyperbolas, meet one another, a ſtraight line paſſing through the point of concourſe, and biſect-

ing

BOOK III. ing the line joining the points of contact, will be a diameter. The truth of this is evident from Cor. Prop. VIII. Book II. and the above.

Cor. 2. From the above it is evident, that if A C be a double ordinate to B D, a diameter of any conic section A B C, and if A E, touching the section in A, meet the diameter in E, then if the straight line E C be drawn, it will touch the section in C.

DEFINITIONS.

VII.

Fig. 84. If from B, the vertex of the axis A B of the parabola P B M, a segment B F be taken in the axis equal to one fourth of the parameter of the axis, the point F is called the *Focus*, or *Umbilicus*, of the parabola.

Cor. The double ordinate T S, drawn through F the focus of any conic section, is equal to the parameter of the axis passing through the focus. This has been proved in the ellipse and hyperbola in Cor. 4. to the eleventh Definition in Book II. In the parabola the square of T F is equal to the rectangle under B F, and four times B F, by this Def. and Prop. III. and therefore 4 TF^2 is equal to 4 B F × 4 B F. But (4. ii.) 4 TF^2 is equal to TS^2, and consequently T S is equal to 4 B F, and therefore, by this Def. equal to the parameter of the axis A B.

VIII.

As in the ellipse and hyperbola, so in the parabola, the straight line touching the section in T, the extremity of the double ordinate drawn as above, is called the *Focal Tangent* to the parabola.

IX.

The transverse axis of an ellipse or hyperbola, and the axis of a parabola, is sometimes called the *Focal Axis* of the section.

X. If

X.

If the focal tangent T G, belonging to the focus F in any conic ſection P B M, meet the focal axis A B in X, the ſtraight line X Y at right angles to A B is called a *Directrix* of the ſection. And, if in the ellipſe or hyperbola O be the other focus, and the focal tangent belonging to O meet the focal axis A B in K, the ſtraight line K L at right angles to A B is alſo called a directrix of the ellipſe or hyperbola. Fig. 84. 85. 86.

Cor. 1. As in the ellipſe and hyperbola the foci F, O are equally diſtant from C the center, it is evident from the above, and Prop. VII. Book II. that the directrices X Y, K L are equally diſtant from the center.

Cor. 2. In the parabola, the focus F and the directrix X Y are equally diſtant from B, the vertex of the axis, by the above and Prop. V.

PROP. VII.

If a tangent paſſing through the vertex of the focal axis of a conic ſection meet a focal tangent, its ſegment between the point of contact and point of concourſe will be equal to the ſegment of the axis between the point of contact and the focus to which the focal tangent belongs.

Let the tangent B G, paſſing through B the vertex of the focal axis A B, of any conic ſection P B M, meet in the point G the focal tangent T G belonging to the focus F; the ſegment G B is equal to the ſegment F B. And in the ellipſe and hyperbola the tangent A H, paſſing through A the other vertex of the focal axis, and meeting the focal tangent T G in H, is equal to the ſegment A F. Fig. 84. 85. 86.

As far as this Propoſition relates to the ellipſe, or hyperbola, it has been proved in Prop. XII. Book II.

In

BOOK III. In the parabola, every thing remaining as in the seventh Definition and its Cor. T F is double of F B, and therefore, by Cor. 2. to the tenth Definition, T F is equal to F X. Again, by Prop. II. T F, G B are parallel, and therefore (4. vi.) T F : F X : : G B : B X, and G B is equal to B X, and consequently equal to F B.

PROP. VIII.

A straight line drawn from any point in the curve of a conic section to a focus is to a straight line drawn from the same point perpendicular to the directrix nearest this focus, as the segment of the axis between the same focus and the nearest vertex, to the segment between this vertex and the directrix: and, in the ellipse and hyperbola, a straight line drawn from the same point in the curve to the other focus is to a straight line drawn from the same point perpendicular to the other directrix in the same ratio.

Fig. 84. 85. 86. A straight line P F, drawn from any point P in the curve of the conic section P B M to the focus F, is to P Y perpendicular to X Y, the directrix nearest to F, as the segment F B, of the focal axis between F and the vertex B, to the segment B X of the same axis between B the vertex and the directrix: and, in the ellipse and hyperbola, P O drawn to the other focus O is to P Q drawn perpendicular to the other directrix K L in the same ratio.

For, the rest remaining as in the preceding Prop. and Def. X. through P draw P M an ordinate to the axis A B, and let it meet the curve again in M, the axis A B in R, and the focal tangent T G in N. Then, by Prop. XIII. Book I. $TG^2 : TN^2 : : GB^2 : PN \times NM$. But, by Prop. II. N M, T F, G B are parallel, and therefore (10. and 22. vi.) $TG^2 : TN^2 : : FB^2 : FR^2$; and therefore

(11.

(11. v.) $FB^2 : FR^2 :: GB^2 : PN \times NM$. Consequently, as by Prop. VII. FB is equal to GB, FB^2 is equal to GB^2, and (14. v.) FR^2 is equal to $PN \times NM$; and therefore (6. ii. and 47. i.) RN^2 is equal to PF^2, and RN is equal to PF. But (4. vi.) RN : RX :: GB : BX, and therefore as (34. i.) PY is equal to RX, and, by Prop. VII. FB is equal to GB, PF : PY :: FB : BX.

Again, in the ellipfe and hyperbola, the reft remaining as above, let C be the center, and let NV perpendicular to the directrix KL meet KL in V. Then (34. i.) NV, PQ are equal, and VK is equal to NR, and confequently equal to PF. Let CD the femiconjugate axis be produced till it meet the focal tangent TG in I, and, by Cor. 2. Prop. XIII. Book II. CI will be equal to CB or CA. Alfo (4. vi.) XC : CI :: XK : KL; and therefore, as XC, CK are equal, KL is equal to AB the tranfverfe axis. Confequently, by Cor. 1. Prop. XIII. Book II. LV is equal to PO; and as LV is parallel to GB, and VN to BX, LV : VN :: GB : BX. On account of the equals therefore, PO : PQ :: FB : BX.

Cor. 1. In the ellipfe and hyperbola, (4. vi.) CI : CX :: NR : RX; and therefore on account of the equals CB : CX :: PF : PY. But, by Prop. VII. Book II. CB : CX :: CF : CB; and therefore (11. v.) CF : CB :: PF : PY. For the fame reafons, or by the above, (and 11. v.) CF : CB :: FB : BX.

Cor. 2. By the preceding Cor. in the ellipfe PF is lefs than PY, but in the hyperbola PF is greater than PY. And, as by the above (and 11. v.) PF : PY :: PO : PQ, in the ellipfe PO is lefs than PQ, but in the hyperbola PO is greater than PQ.

Cor. 3. A ftraight line drawn from any point in the curve of a parabola to the focus is equal to a ftraight line Fig. 84.

BOOK III. line drawn from the ſame point perpendicular to the directrix. For P F : P Y : : F B : B X, and F B is equal to B X.

SCHOLIUM.

Some writers on conic ſections have choſen this property as the primary one for their treatiſes, and according to it have defined the ſections in the following manner.

Fig. 84. 85. 86. Let F be a point without the ſtraight line X Y, and whilſt a ſtraight line F P revolves about F as a center, let a point P ſo move in F P that F P may always be to P Y, perpendicular to X Y, in a given ratio. The curve deſcribed by the point P will be a conic ſection; and it will be a parabola, ellipſe, or hyperbola, according as F P is equal to, leſs, or greater than P Y.

PROP. IX.

If from any point in the curve of a parabola a ſtraight line be drawn to the focus, and a ſtraight line perpendicular to the directrix, the angle contained by theſe ſtraight lines will be biſected by a tangent paſſing through the ſame point.

Fig. 89. From the point P in the curve of the parabola P B R let the ſtraight line P F be drawn to F the focus, and the ſtraight line P D perpendicular to D X the directrix; the ſtraight line P E, touching the parabola in P, biſects the angle F P D.

For let the tangent P E meet the axis A B in E, and let P A be an ordinate to A B. Then, by Prop. V. A B is equal to B E; and as, by Cor. 2. to the tenth Definition, F B is equal to B X, A X is therefore equal to F E. But (34. i.) A X is equal to P D; and, by Cor. 2. Prop. VIII. P D is equal to P F. Conſequently P F,

F E

F E are equal, and therefore (5. i.) the angle F P E is equal to the angle F E P. But as P D, A E are parallel, the angle F E P (29. i.) is equal to the angle D P E. Consequently the angle F P E is equal to the angle D P E, and therefore the angle F P D is bisected by the tangent P E.

Cor. If a straight line touching a parabola meet the axis, the segment of the axis between the point of concourse and the focus is equal to the straight line drawn from the point of contact to the focus. This is evident from the above, for F E is equal to P F.

SCHOLIUM.

It is not certain when, or by whom, the name *Focus*, or *Umbilicus*, was first given to a point in a conic section. Neither of the two occurs in the Treatise of Apollonius, or in the writings of Archimedes, who occasionally mentions properties of the sections. The points themselves, however, in the ellipse and hyperbola, were well known to Apollonius. He calls them *puncta ex applicatione facta*; and he demonstrates the most important properties of lines related to them. He does not mention the focus of the parabola.

It is highly probable, from analogy of general principles, and from the history of this branch of science as far as it has been traced, that optical pursuits first suggested each of the two names. For, in opticks, if a ray of light fall upon a plane surface, the angle of incidence is equal to the angle of reflexion; and if a ray of light fall upon a curve, and a straight line touch the curve in the same point, the angle contained by the incident ray and tangent will be equal to the angle contained by the reflected ray and tangent, as a tangent is the direction of a curve in the point of contact.

These

BOOK III.

Fig. 89. 90. 91.

Theſe truths being premiſed, ſuppoſe a conic ſection R B P to revolve about A B, the focal axis, and that a concave ſpeculum is formed by the curve by this revolution. Let the ſtraight line T P E touch any one of the ſections in the point P. In the ellipſe and hyperbola let F, O be the foci, and let F be the focus of the parabola.

Fig. 91.

1. Let K P be a ray of light, whoſe direction, in a ſtraight line, paſſes through O the focus oppoſite to F, within the hyperbolic ſpeculum R B P. Let it fall upon the ſpeculum in the point P, and draw P F, and produce K P to O. Then, as by Prop. XV. Book II. the angle F P E is equal to the angle O P E, and conſequently (15. i.) equal to the angle K P T, the ſtraight line P F will repreſent the ray of light after reflexion. For, if a perpendicular to T E be drawn from P, the angle contained by it and K P will be equal to the angle contained by it and F P; the angle of incidence being equal to the angle of reflexion. Hence if any number of rays fall on a concave hyperbolic ſpeculum, and be converging to the focus in the oppoſite hyperbola, they will be reflected to the focus within the ſpeculum.

Fig. 90.

2. Let A R D B P be a concave elliptic ſpeculum. Let a ray of light proceeding from the focus O fall upon the ſpeculum at P, and let P F be drawn. Then will P F repreſent the ray after reflexion, for the ſame reaſons as above, as the angles O P T, F P E are equal, by Cor. 1. to Prop. XV. Book II. Hence it is evident, that if any number of rays proceed from one focus of an elliptic concave ſpeculum, they will be reflected into the other. As rays of heat are ſubject to the ſame laws with rays of light, as to incidence and reflexion, if a fire or any heated body be placed at O, within the concave elliptic ſpeculum, the whole of the heat after reflexion will meet at F. Perhaps attention

to

to this property might be of confiderable ufe in fitting up fire-places, reverberating furnaces, &c. BOOK III.

3. Let K P be a ray of light parallel to A B the axis of the parabola, by whofe revolution the parabolic fpeculum is generated, and let P F be drawn. Then will P F reprefent the ray after reflexion. For, K P being produced to D, the angle F P E will be equal to the angle D P E, by the Propofition preceding this Scholium, and therefore (15. i.) the angle K P T will be equal to the angle F P E. Confequently, as above, P F is the ray after reflexion. Hence if any number of rays, parallel to the axis, fall upon a concave parabolic fpeculum they will all after reflexion meet in the focus. Fig. 89.

The very confiderable magnifying powers which reflecting telefcopes are capable of, with parabolic fpecula, are to be attributed to this property. For a celeftial body being at an immenfe diftance, the rays which iffue from it upon the parabolic fpeculum are, as to fenfe, parallel to the axis; and, being all reflected to the focus, a diftinct and vivid image of the body is produced, provided the compofition of the metal be good and the parabolic figure juft.

PROP. X.

A ftraight line drawn from the focus of a parabola, perpendicular to a tangent, is a mean proportional between the ftraight line drawn from the point of contact to the focus, and the fegment of the axis between the focus and the vertex of the axis.

From F the focus of the parabola P B R let the ftraight line F G be drawn perpendicular to the ftraight line P E, touching the parabola in P, and draw P F; F G is a mean proportional between P F and F B, the fegment Fig. 89.

 of

BOOK III.

of the axis A B, between F and B the vertex of the axis.

For let the tangent P E meet the axis in E, and draw B G, and let P A be an ordinate to the axis. Then, by Cor. Prop. IX. P F, F E are equal, and therefore (5. i.) the angles F P E, F E P are equal. Confequently in the triangles F G P, F G E, as the angles at G are right angles, P G is (26. i.) equal to G E, and the angles P F G, E F G are equal. Confequently, as, by Prop. V. A B is equal to B E, P G : G E : : A B : B E, and (2. vi.) G B is parallel to the ordinate P A, and therefore G B F is a right angle. The triangles P F G, G F B are therefore equiangular, and (4. vi.) P F : F G : : F G : F B.

The above Prop. is Lemma XIV. Lib. I. of the Principia.

Cor. 1. Hence (Cor. 2. 20. vi.) $PF^2 : FG^2 :: PF : FB$.

Cor. 2. The concourfe of any tangent P E with a ftraight line F G, drawn from the focus of the parabola perpendicular to the tangent, is in the ftraight line B G, which touches the parabola in the vertex of the axis. For, by the above, G B is parallel to the ordinate P A, and therefore the Cor. is evident by Prop. II.

Cor. 3. If a ftraight line touch a parabola, and cut a ftraight line drawn from the focus to the directrix at right angles, it will bifect it. For, the reft remaining as above, let F G produced meet the directrix in D. Then as G B, D X are perpendicular to the axis, they are parallel, and (2. vi.) F B : B X : : F G : G D, and as, by Cor. 2. Def. X. F B is equal to B X, F G is equal to G D.

SCHOLIUM.

If a ftraight line pafs through a point moving in the curve of a conic fection, and always touch the fection, and

and if a straight line revolve about a focus of the section as a center, and be always perpendicular to the moving tangent, the magnitude of the perpendicular will be less varied in the hyperbola than in the parabola, but it will be more varied in the ellipse than in the parabola. BOOK III.

For let the point P be supposed to move in the curve B P of the conic section P B R, and let the straight line P E accompany it in its motion, and always touch the section. Let the straight line F G revolve about F, a focus of the section, and let it be always perpendicular to the tangent P E. Fig. 89. 90. 91.

In the ellipse and hyperbola let C be the center, C D the semiconjugate axis, and C H the semidiameter parallel to the tangent P E. Then, by Prop. XIX. Book II. $FG : FP :: CD : CH$, and therefore, by Lemma V. $FG^2 : FP^2 :: CD^2 : CH^2$. But O being the other focus, and P O being drawn, by Prop. XVIII. Book II. $CH^2 = FP \times PO$, and therefore $FG^2 : FP^2 :: CD^2 : FP \times PO$, and $FG^2 = \frac{FP^2 \times CD^2}{FP \times PO} = \frac{FP \times CD^2}{PO}$. For the same reasons if p denote another position of the moving point, Fg the perpendicular at that position, and Fp, pO straight lines drawn from p to the foci, then $Fg^2 = \frac{Fp \times CD^2}{pO}$. Consequently $FG^2 : Fg^2 :: \frac{FP \times CD^2}{PO} : \frac{Fp \times CD^2}{pO} :: \frac{FP}{PO} : \frac{Fp}{pO}$. Fig. 90. 91.

If in the parabola p denote another position of the moving point, Fg the perpendicular at that position, and pF a straight line drawn to the focus, then, by the Proposition preceding this Scholium, (and 17. vi.) $Fg^2 = Fp \times FB$, and $FG^2 : Fg^2 :: FP \times FB : Fp \times FB :: FP : Fp$; or $FG^2 : Fg^2 :: \frac{FP}{FB} : \frac{Fp}{FB}$. Fig. 89.

BOOK III.

In the hyperbola and ellipſe, therefore, the ſquare of the perpendicular F G varies as the value of the fraction $\frac{FP}{PO}$ varies, and in the parabola it varies as $\frac{FP}{FB}$ varies.

But if the numerator and denominator of a fraction be each variable, then if they always increaſe or decreaſe in the ſame proportion, the value of the fraction will be always the ſame. Thus if the fraction be $\frac{Q}{R}$, and if while Q varies and becomes q, R varies and becomes r, and if it be $Q : R :: q : r$, then $\frac{Q}{R} = \frac{q}{r}$, by converting the proportion into an equation. From hence it is alſo evident, that the more nearly the numerator and denominator increaſe or decreaſe in the ſame proportion, the leſs will the value of the fraction be varied; but, on the contrary, the more they differ from a proportional increaſe or decreaſe, the more will the value of the fraction be varied. Now in the hyperbola the difference between F P, P O is, in every ſituation of P, equal to A B the tranſverſe axis, and therefore, in every inſtant they vary, they receive an equal increaſe or diminution; but however, in the parabola, F P may increaſe or diminiſh, F B remains conſtant. In the hyperbola therefore the value of the fraction $\frac{FP}{PO}$ varies leſs than the value of the fraction $\frac{FP}{FB}$ in the parabola. Again, in the ellipſe the ſum of P F, P O is equal to A B the tranſverſe axis, and therefore if either of them increaſe, the other will diminiſh; and conſequently in varying they will differ more from a proportional increaſe at the ſame time, or decreaſe at the ſame time, than F P, F B in the parabola. In the ellipſe

lipse therefore the value of the fraction $\frac{FP}{PO}$ varies more than the value of the fraction $\frac{FP}{FB}$ in the parabola.

In the hyperbola therefore the square of F G varies less, but in the ellipse it varies more, than it varies in the parabola; and consequently F G varies less in the hyperbola, but more in the ellipse, than it varies in the parabola. See the Principia, Cor. 6. Prop. XVI. Lib. I.

PROP. XI.

If from a point in which a straight line touches a parabola two straight lines be drawn to the axis, one of them an ordinate to it, and the other at right angles to the tangent, the segment of the axis intercepted between them will be equal to half the parameter of the axis: and if a straight line touching a parabola meet the axis, and a straight line, at right angles to the tangent, be drawn from the point of contact to the axis, the segment of the axis intercepted between them will be equal to half the parameter of the diameter passing through the point of contact.

Part I. From the point P, in which the straight line P E touches the parabola P B R, let the two straight lines P A, P H be drawn to A B the axis, one of them P A an ordinate to it, and the other P H at right angles to the tangent P E; the segment H A of the axis, intercepted between them, is equal to half the parameter of the axis. Fig. 89.

For let the tangent P E meet the axis in E, and then as P A is an ordinate to the axis, it is at right angles to H E; and (Cor. 8. vi.) the square of P A is equal to the rectangle under E A, A H. But, by Prop. V. E B, B A are equal, and, by Prop. III. the square of P A is equal

 to

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to the rectangle under B A and the parameter of the axis. Consequently the rectangle under E A, A H is equal to the rectangle under B A and the parameter of the axis, and therefore (16. vi.) E A is to B A as the parameter of the axis to A H; and as B A is half of E A, A H must be half of the parameter of the axis.

Part II. Let the straight line P E touching the parabola P B R in P meet the axis A B in E, and let the straight line P H, at right angles to P E, meet the axis A B in H; the segment E H, intercepted between P E, P H, is equal to half the parameter of the diameter P K, passing through the point of contact.

For, the rest remaining as in the preceding part, the square of P E (Cor. 8. vi.) is equal to the rectangle under H E, E A; and, by Cor. 1. Prop. III. the square of P E is equal to the rectangle under E B and the parameter of P K. Consequently the rectangle under H E, E A is equal to the rectangle under E B and the parameter of P K, and therefore (16. vi.) E A is to E B as the parameter of P K to H E; and as E B is half of E A, H E must be equal to the half of the parameter of P K.

Cor. 1. The parameter of the axis is less than the parameter of any other diameter.

Cor. 2. A straight line drawn from any point in the curve of a parabola to the focus is equal to a fourth part of the parameter of the diameter passing through the same point. If the point be the vertex of the axis, this is evident from the seventh Definition; but for any other point P let every thing remain as in this Prop. and draw P F to the focus F. Then, by Cor. Prop. IX. F P, F E are equal; and as E P H is a right angle, if with F as a center, and F P as a distance, a circle be described, it will pass (31. iii.) through E and H. Consequently E F, F P, F H are equal, and therefore, by Part II. of this Prop. each of them is equal to a fourth

part

part of the parameter of P K. This is Lemma XIII. Lib. I. of the Principia. BOOK III.

Cor. 3. The diftance of the vertex of any diameter of a parabola from the directrix is equal to a fourth part of the parameter of the diameter. This is evident from the preceding Cor. and Cor. 3. Prop. VIII.

PROP. XII.

Any ftraight line drawn through the focus of a parabola, and terminated both ways by the curve, is equal to the parameter of the diameter to which it is a double ordinate.

Let the ftraight line A H, paffing through F the focus of the parabola A B C, meet the curve in A and H, and be a double ordinate to the diameter D E; A H is equal to the parameter of D E. Fig. 87.

If D E be the axis, the Prop. is the fame as the Cor. to the feventh Definition. Let D E therefore not be the axis, and let it meet A H in E. Let D G touch the parabola in D the vertex of D E, and meet B F the axis in G, and draw D F. Then, by Cor. 1. to the firft Definition, and Prop. II. G E is a parallelogram, and therefore (34. i.) D E, G F are equal. But, by Cor. Prop. IX. G F, D F are equal, and, by Cor. 2. Prop. XI. D F is equal to a fourth part of the parameter of D E. Confequently D E is equal to a fourth part of the parameter of D E, and therefore, by Prop. III. AE^2 is equal to $DE \times 4DE$, and $4AE^2$ is equal to $4DE \times 4DE$. But (4. ii.) $4AE^2$ is equal to AH^2, and therefore A H is equal to 4 D E, and confequently equal to the parameter of D E.

Cor. If a ftraight line in a parabola pafs through the focus, and cut the diameter to which it is an ordinate, the abfcifs of the diameter will be equal to the diftance

BOOK III. of its vertex from the focus, and alſo from the directrix. For, as above, D E is equal to D F, and therefore, by Cor. 3. Prop. VIII. the remaining part of this is evident.

DEFINITIONS.

XI.

Fig. 88. The ſuperficies A C L B I A, bounded by the ſtraight line A C and the curve A I B L C of a parabola, is called an *interior parabolic ſegment*; and if the ſtraight lines A E, C E, touching the parabola in A, C, meet one another in E, the ſuperficies bounded by A E, C E, and the curve A I B L C, is called the *exterior parabolic ſegment*, correſponding to the interior ſegment firſt mentioned.

XII.

If the ſtraight line R S, parallel to A C, touch the parabola in B, and meet in the points R, S the diameters A R, C S, the parallelogram A R S C is ſaid to be circumſcribed about the interior parabolic ſegment A I B L C A.

PROP. XIII.

In an interior parabolic ſegment a rectilineal figure may be inſcribed, and a correſponding rectilineal figure may be inſcribed in the exterior ſegment, ſo that the rectilineal figure in the interior ſegment ſhall be double of that inſcribed in the exterior; and theſe rectilineal inſcribed figures may be ſuch, that each ſhall be leſs than the ſegment in which it is inſcribed by a ſuperficies leſs than any given ſuperficies.

Fig. 88. Let A C L B I A be an interior, and E A I B L C E the correſponding exterior parabolic ſegment, as in the eleventh Definition; a rectilineal figure may be inſcribed in the ſegment A C L B I A, and a correſponding recti-

rectilineal figure may be infcribed in the fegment E A I B L C E, fo that the rectilineal figure in the interior fegment fhall be double of that in the exterior; and thefe rectilineal figures may be fuch, that each fhall be lefs than the fegment in which it is infcribed by a fuperficies lefs than a given fuperficies O.

For bifect A C in D, and draw E D, and let it cut the curve in B. Let the ftraight line F G touch the parabola in B, and meet the tangent E A in F, and the tangent E C in G, and draw A B, C B. Then, by Cor. 1. Prop. VI. E D is a diameter of the parabola, and, by Prop. II. F G, A C are parallel; and, by Prop. V. D B, B E are equal. Confequently (2. vi.) A F, F E are equal, and (1. vi.) the triangle A F B is equal to the triangle E F B; and therefore, as (1. vi.) the triangle A B D is equal to the triangle A E B, the triangle A B D is double of the triangle E F B. For the fame reafons the triangle C B D is double of the triangle E G B; and therefore the triangle A B C, infcribed in the interior parabolic fegment, is double of the triangle E F G, infcribed in the exterior parabolic fegment. Again, bifect A B in H, and draw F H, and let it meet the curve in I. Let the ftraight line M N touch the parabola in I, and meet the tangent A F in M, and the tangent F B in N, and draw A I, I B. Then, as above, it may be proved, that the triangle A I B, infcribed in the interior parabolic fegment A I B A, is double of the triangle F M N, infcribed in the exterior parabolic fegment F A I B F. Alfo bifect B C in K, and draw G K, and let it meet the curve in L. Let the ftraight line P Q touch the parabola in L, and meet the tangent C G in Q, and the tangent B G in P, and draw B L, L C. Then it may be alfo proved, as above, that the triangle B L C, infcribed in the interior parabolic fegment B L C B, is double of the

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triangle G P Q, inscribed in the exterior parabolic segment G B L C G. Consequently the rectilineal figure A I B L C, inscribed in the interior parabolic segment first mentioned, is double of the rectilineal figure E M N P Q, inscribed in the corresponding exterior parabolic segment first mentioned. In the same manner the inscription in each segment may be continued, so that the whole figure inscribed in the interior segment shall be double the whole figure inscribed in the exterior segment.

The inscription may also be continued till the rectilineal figure shall be less than the segment in which it is inscribed by a superficies less than the superficies O. For in the interior segment the triangle A B C, being equal to half the triangle E A C, is greater than half the segment A I B L C A; and for the same reasons the triangle A I B is greater than half the segment A I B A, and so of other remaining segments. Also in the exterior segment the triangle E F G, being half of the rectilineal figure E A B C E, is greater than half the segment E A I B L C E; and for the same reasons the triangle F M N is greater than half the segment F A I B F, and so of other remaining segments. Consequently (1. x.) the inscription may be continued till each of the rectilineal figures shall be less than the segment in which it is inscribed by a superficies less than the given superficies O.

Cor. The tangent F G being produced till it meet the diameters A R, C S in R and S, the circumscribed parallelogram R A C S is equal to the triangle E A C, and therefore equal to the interior and exterior parabolic segments taken together. For, by the above, E F, F A are equal, and as R A, E B are parallel, the triangles E F B, A F R are equiangular, and (26. and

4. i.)

4. i.) therefore equal. For the ſame reaſons the triangles E G B, C G S are equal; and conſequently the Cor. is evident.

PROP. XIV.

An interior parabolic ſegment is double of the correſponding exterior parabolic ſegment; and the interior parabolic ſegment is to the circumſcribed parallelogram as two to three.

Every thing remaining as in the preceding Prop. and its Cor. the interior parabolic ſegment A C L B I A is double the correſponding exterior parabolic ſegment E A I B L C E, and the interior ſegment is to the circumſcribed parallelogram R A C S as two to three. Fig. 88.

Part I. If the interior ſegment be not double of the exterior ſegment, it muſt either be greater or leſs than its double. Firſt, let it be greater than the double of the exterior ſegment, and let the ſuperficies O be equal to the exceſs of the interior ſegment above the double of the exterior. Let correſponding rectilineal figures be inſcribed in the interior and exterior ſegments, as in the preceding Prop. and let them be ſuch that the rectilineal figure A I B L C, inſcribed in the interior ſegment, may be leſs than the ſegment in which it is inſcribed by a ſuperficies leſs than O, and let E M N P Q E be the correſponding rectilineal figure inſcribed in the exterior ſegment. Then the rectilineal figure A I B L C is greater than the double of the exterior ſegment, and therefore its half is greater than the exterior ſegment. But the rectilineal figure E M N P Q E is half of A I B L C, by Prop. XIII. and conſequently the rectilineal figure E M N P Q E is greater than the parabolic ſegment in which it is inſcribed: which is abſurd. The interior ſegment therefore cannot be greater than the double of the

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the exterior ſegment. Secondly, let the interior ſegment be leſs than the double of the exterior ſegment, and conſequently the exterior ſegment greater than half the interior ſegment. Let the exceſs of the exterior ſegment above half the interior be equal to the ſuperficies O; and let correſponding rectilineal figures be inſcribed in each ſegment, as in the preceding Prop. ſo that the rectilineal figure E M N P Q E inſcribed in the exterior ſegment may be leſs than the ſegment in which it is inſcribed by a ſuperficies leſs than O. Then the rectilineal figure E M N P Q E is greater than half the interior ſegment, and therefore its double is greater than the interior ſegment. Let A I B L C A, E M N P Q E be the correſponding rectilineal figures inſcribed in the two ſegments; and then, by Prop. XIII. as A I B L C A is double of E M N P Q E, it is greater than the interior ſegment in which it is inſcribed: which is abſurd. The interior ſegment is therefore not leſs than the double of the exterior ſegment. Conſequently as the interior ſegment is neither greater nor leſs than the double of the exterior, it is equal to the double of the exterior ſegment.

Part II. As, by Part I. the interior ſegment is double of the exterior, the interior ſegment is to the exterior as two to one, and therefore (18. v.) the interior ſegment is to the triangle E A C as two to three. Conſequently, by Cor. Prop. XIII. the interior ſegment is to the circumſcribed parallelogram R A C S as two to three *.

* Archimedes was the firſt who proved that a parabolic ſegment is equal to two thirds of the circumſcribed parallelogram. Of this truth he gave two demonſtrations. The firſt may be conſidered as mechanical, as it depends upon the primary properties of the lever. The ſecond is ſtrictly geometrical, and may be eaſily underſtood from the above; for he proves, that the triangle A B D is equal to four times the

DEFINITIONS.

XIII.

Fig. 92.

If A M N be the vertical plane to the oppoſite hyperbolas E V F, G L H, as in the eighteenth Definition of the firſt Book, and cut the cone in the ſides A M, A N, and if planes A K, A I, touching the cone in the ſides A N, A M, cut the plane of the hyperbolas in the ſtraight lines K Q, I R, the ſtraight lines K Q, I R are called the *Aſymptotes* of either of the hyperbolas, or of the oppoſite hyperbolas.

Cor. 1. As the vertical plane A M N is parallel to the plane of the hyperbolas E V F, G L H, the aſymptotes K Q, I R (16. xi.) are parallel to A N, A M, ſides of the cone.

Cor. 2. The aſymptotes K Q, I R do not meet the curve of either of the oppoſite hyperbolas. For the planes A K, A I touch the oppoſite cones in the ſides A N, A M, and as the aſymptotes are parallel to theſe ſides, they do not meet either of the oppoſite conical ſuperficies. The Cor. is therefore evident.

Cor. 3. Any two ſtraight lines T S, P O drawn in the planes A K, A I from the aſymptotes to the ſides A N, A M, and parallel to the baſe of the cone, are equal, and touch the conical ſuperficies. For let N M, I K, N K, M I be the lines of common ſection of the baſe with the vertical plane, the plane of the hyperbolas,

the triangle A I B, each of theſe triangles being inſcribed according to the conditions ſtated above. The remainder of his demonſtration is then equivalent to this, (ſee Maclaurin's Algebra, §. 68.) that the ſum of the infinite ſeries $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} +$ &c. can neither be greater or leſs than $\frac{4}{3}$, the triangle A B C being analagous to 1.

The above Propoſition and the foregoing remarks being clearly comprehended, the 5th Cor. to Lemma XI. of Sir Iſaac Newton's Doctrine of Prime and Ultimate Ratios will be eaſily underſtood.

and

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and the planes A K, A I. Then, as Q K, S N are parallel, and as S T, being parallel to the baſe, is parallel to N K, the ſtraight line S T (34. i.) is equal to N K. For the ſame reaſons O P is equal to M I. Alſo N K, M I are equal. For if N K, M I be parallel, then M K is a parallelogram, and (34. i.) N K is equal to M I. But if N K be not parallel to M I, let them meet in W; and as they are in the tangent planes, they will touch the circle M D N, the baſe of the cone. The ſtraight lines W N, W M (36. iii.) are therefore equal, and (2. vi.) W N : N K : : W M : M I. Conſequently (14. v.) N K is equal to M I, and therefore S T, O P are equal, and as they are in the planes A K, A I, they touch the conical ſuperficies.

XIV.

The angle I C K, or Q C R, within which either of the oppoſite hyperbolas is ſituated, is called the *interior angle of the aſymptotes*; and the angle R C K, or Q C I, adjacent to it, is called the *exterior angle of the aſymptotes.*

PROP. XV.

The point in which the aſymptotes of an hyperbola cut one another is the center of the hyperbola; and any ſtraight line paſſing through the center and falling within the interior angle of the aſymptotes is a tranſverſe diameter; but any ſtraight line paſſing through the center and falling within the exterior angle of the aſymptotes is a ſecond diameter of the hyperbola.

Fig. 93.

Part I. Let A B D, E F G be oppoſite hyperbolas, and let their aſymptotes I P, R N cut one another in C; C is the center of the hyperbola, or oppoſite hyperbolas.

In the curve of the hyperbola A B D take any two points A, D, and draw the ſtraight line A D. Then A D will

c.

Fig. 87.

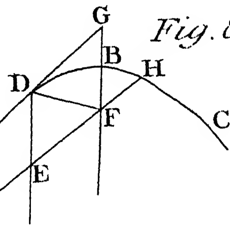


Fig. 82.

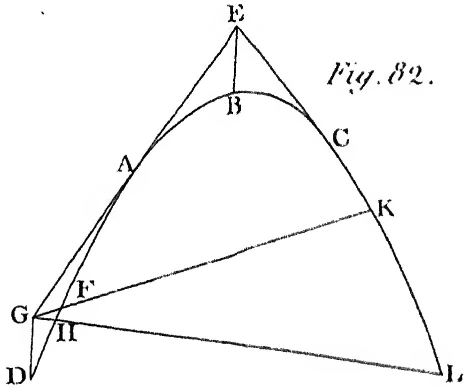
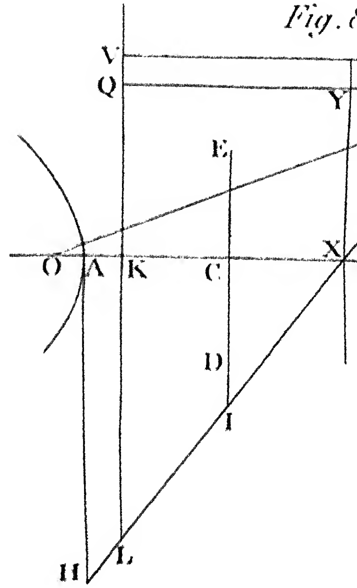


Fig. 8



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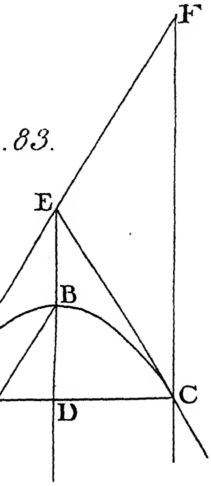
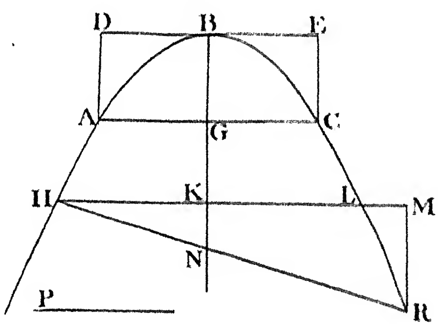


Fig. 81.



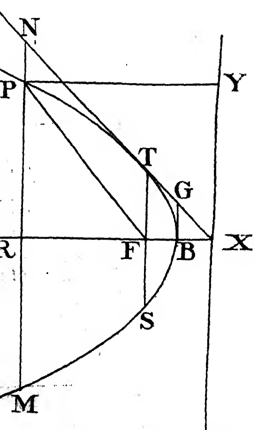


Fig. 85.

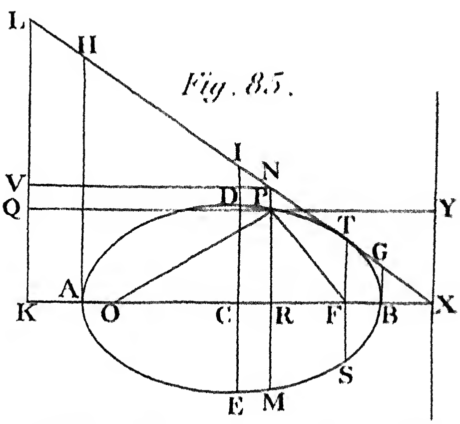
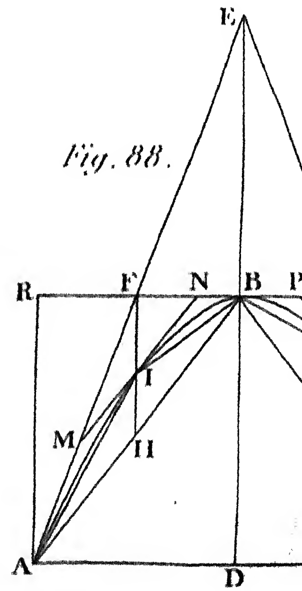


Fig. 88.



will meet both the asymptotes; for if it were parallel to either of the two, it would meet the curve in one point only, by Cor. 1. to the thirteenth Definition, and Prop. XIV. Book I. Let A D meet the asymptote R N in N, and the other in P; and let L M, parallel to A D, touch the hyperbola A B D in B, and meet R N in L, and I P in M. Let N V, L X, M Y, P W be straight lines parallel to the base of the cone in which the hyperbola was formed, and let them be supposed to have touched the conical superficies in the points V, X, Y, W; and then, by Cor. 3. to the thirteenth Definition, N V, L X, M Y, P W are equal. Also, by Cor. 1. Prop. XI. and Cor. 1. Prop. X. Book I. $BM^2 : MY^2 :: BL^2 : LX^2$; and $AP \times PD : PW^2 :: DN \times NA : NV^2$. Consequently (14. v.) BM^2 is equal to BL^2, and $AP \times PD$ is equal to $DN \times NA$; and therefore B M is equal to B L, and, by Lemma VI. P D is equal to N A. Draw C B, and let it meet A D in O; and then (4. vi.) C B : C O :: B L : O N :: B M : O P, and therefore (14. v.) O N is equal to O P. Consequently C B bisects A D, and for the same reasons it will bisect any straight line parallel to A D in the hyperbola; and therefore, by Cor. 1. Prop. III. Book II. C B is a diameter. Next, let the point H be in the curve of one hyperbola, and F in the other. Draw F H, and let it meet R N in Q, and I P in M. Let M Y, Q Z be two straight lines parallel to the base of the cone in which the sections were formed, and let them be supposed to have touched the conical superficies in Y and Z. Then, by Cor. 1. Prop. X. Book I. $HM \times MF : MY^2 :: HQ \times QF : QZ^2$, and therefore, by Cor. 3. to the thirteenth Definition (and 14. v.) $HM \times MF$ is equal to $HQ \times QF$, and, by Lemma VI. M H is equal to Q F. Consequently if C S be drawn bisecting Q M in S, it will bisect F H; and in the same manner, as above, it may be proved,

BOOK III.

proved, that C S will biſect any other ſtraight line G K parallel to F H, and terminated by the oppoſite curves. By Cor. 1. Prop. III. Book II. C S is therefore a diameter, and conſequently C is the center of the hyperbola.

Fig. 92. Part II. Every thing remaining as in the thirteenth Definition, any ſtraight line L V paſſing through the center C, and falling within the interior angle I C K of the aſymptotes, is a tranſverſe diameter of the hyperbola.

For draw A C, and through A C, L C V let a plane paſs, and let its line of common ſection with the vertical plane be A Z. Then (16. xi.) A Z, C V are parallel, as are alſo A N, C K; and therefore (10. xi.) the angles Z A N, V C K are equal. In the ſame manner it may be demonſtrated that the angles M A Z, I C V are equal, and that the angle M A N is equal to the angle I C K. The ſtraight line A Z therefore, paſſing through A the vertex of the cone, falls within the oppoſite ſuperficies, and conſequently, as in Part II. Prop. IX. Book I. it may be proved, that L C V meets the oppoſite conical ſuperficies. The ſtraight line L C V therefore, paſſing through C the center, meets the curves of the oppoſite hyperbolas, and conſequently is a tranſverſe diameter.

Fig. 93. Part III. Any ſtraight line C S, paſſing through C and falling within R C P, the exterior angle of the aſymptotes, is a ſecond diameter of the hyperbola.

For if through O, any point within the hyperbola A B D, a ſtraight line as A D be drawn parallel to C S, it will meet the aſymptotes, by Lemma III. and therefore, as it muſt cut the curve in two points, by Cor. 1. Prop. II. Book II. it is a ſecond diameter.

Fig. 93. *Cor.* 1. If a ſtraight line as L M touch an hyperbola and meet the aſymptotes, its ſegments between the point of contact and the aſymptotes will be equal. And

And if a ſtraight line cutting an hyperbola, or oppoſite hyperbolas, meet the aſymptotes, its ſegments between the curve or curves and the aſymptotes will be equal. This is evident from the above; for it was proved that N A is equal to P D, and H M to F Q.

Cor. 2. If through any point, as P, in either aſymptote, a ſtraight line, as K G, be drawn, meeting the oppoſite hyperbolas in K and G; and a ſtraight line, as P A, be drawn, meeting the curve of the hyperbola A B D in D and A; the rectangle under K P, P G will be equal to the ſquare of the ſemidiameter parallel to K G, and the rectangle under A P, P D will be equal to the ſquare of the ſemidiameter parallel to A P. For let E B be the diameter parallel to K G, and, the reſt remaining as above, let the ſtraight line C U be parallel to the baſe of the cone in which the ſection was formed, and let it be ſuppoſed to have touched the conical ſuperficies in U. Then, by Cor. 3. to the thirteenth Definition, P W, C U are equal, and, by Prop. XII. Book I. $KP \times PG : PW^2 :: BC \times CE$ or $CB^2 : CU^2$; and therefore, as PW^2, CU^2 are equal, $KP \times PG$ is equal to CB^2. Again, by Prop. V. Book II. $KP \times PG$ is to $AP \times PD$ as CB^2 to the ſquare of the ſemidiameter parallel to A P; and therefore (14. v.) the Cor. is evident.

Cor. 3. A ſtraight line as L M, touching the hyperbola A B D in B, and meeting the aſymptotes in L, M, is parallel and equal to the ſecond diameter conjugate to the tranſverſe diameter E B, paſſing through the point of contact. For it may be proved, as in the laſt Cor. that the ſquare of B M is equal to the ſquare of the ſemidiameter parallel to it. Conſequently, by Cor. 1. preceding, and by Cor. 3. Prop. III. Book II. this Cor. is evident.

BOOK III.

PROP. XVI.

According as a transverse diameter of an hyperbola is greater, equal to, or less than its conjugate diameter, the interior angle of the asymptotes is an acute, a right, or an obtuse angle; and any other transverse diameter is greater, equal to, or less than its conjugate diameter.

Fig. 94. Let G B be an hyperbola, of which C is the center, and C K, C I the asymptotes, and let A B be any transverse diameter, and D E the diameter conjugate to it; according as A B is greater, equal to, or less than D E, the interior angle K C I of the asymptotes is an acute, a right, or an obtuse angle; and any other transverse diameter F G is greater, equal to, or less than H L the diameter conjugate to it.

For let N I, touching the hyperbola in B, the vertex of A B, meet the asymptotes in N, I; and let K M, touching the hyperbola in G, the vertex of F G, meet the asymptotes in K, M. Then, by Cor. 1. and Cor. 2. Prop. XV. N I is bisected in B, and K M in G; and N I is equal to D E, and K M equal to H L, and therefore, according as A B is greater, equal to, or less than D E, C B is greater, equal to, or less than B I. But if with B as a center and B I as a distance a circle be described, its circumference will pass through N, and according as C B is greater, equal to, or less than B I, it will pass between C and B, through C, or on the opposite side of C from B. Consequently (by 31. iii. and 21. i.) according as C B is greater, equal to, or less than B I, the angle K C I is an acute, a right, or an obtuse angle. Again, if with G as a center and G M as a distance a circle be described, its circumference will pass through K, and (by 31. iii. and 21. i.) according as the angle K C I is an acute, a right, or an obtuse angle, the circumference of the circle will pass between C and G,

through

through C, or on the oppofite fide of C from G. Confequently, according as K C I is an acute, a right, or an obtufe angle, C G is greater, equal to, or lefs than G M. The Propofition therefore is evident.

If two conjugate diameters of an hyperbola be equal, or if the angle contained by the afymptotes be a right one, it is called an *Equilateral Hyperbola.*

PROP. XVII.

The rectangle under two ftraight lines drawn from a point in the curve of an hyperbola to the afymptotes is equal to the rectangle under two ftraight lines, parallel to them, drawn from any other point in the curve of the fame or oppofite hyperbola to the afymptotes.

The rectangle under the two ftraight lines A E, A D, drawn from the point A in the curve of the hyperbola A V to the afymptotes C H, C K, is equal to the rectangle under the ftraight lines B F, B G, parallel to A E, A D, drawn from the point B in the curve of the fame or oppofite hyperbola to the afymptotes. Fig. 95.

For draw A B, and let it meet the afymptotes in H and K. Then, as A E, B F are parallel, (4. vi.) A E : B F : : H A : H B; and as B G, A D are parallel, B G : A D : : K B : K A. But, by Cor. 1. Prop. XV. H A is equal to K B, and therefore H B is equal to K A. Confequently (11. v.) A E : B F : : B G : A D*, and (16. vi.) A E × A D is equal to B F × B G.

Cor. 1. If A E, B F be parallel to the afymptote C K, and A D, B G be parallel to the afymptote C H; the pa-

* This is the property referred to by writers on Natural Philofophy, when they prove, that the curve formed by the upper furface of a liquid raifed by the force of attraction between two plates, meeting at one end, and kept at a fmall diftance from one another at the other, is an hyperbola.

 rallelograms

BOOK III.

rallelograms E D, F G (14. vi.) are equal. For the angle at C being common to the two parallelograms, they are equiangular, and, by the above, the ſides round the equal angles are reciprocally proportional.

Cor. 2. If from any two points as A, B in the curve of an hyperbola A V B two ſtraight lines A E, B F, parallel to one of the aſymptotes as C K, be drawn to the other aſymptote C H; then C F : C E : : E A : F B. And the ſemitranſverſe diameters C A, C B being drawn, the triangles C F B, C E A are equal.

Cor. 3. If C H, C K be the aſymptotes of an hyperbola A V B, and if from any two points F, E in C H, ſtraight lines F B, E A be drawn parallel to C K, and if F B be drawn to the curve and E A towards it, and if E C be to C F as F B to E A, the point A muſt alſo be in the curve.

DEFINITIONS.

XV.

Fig. 96.

If from C the center of the hyperbola M N Q any two ſemitranſverſe diameters C N, C Q be drawn to the curve, the figure C N Q bounded by the ſemidiameters and the curve N Q is called a *Hyperbolic Sector.*

XVI.

If C R, C G be the aſymptotes of the hyperbola M N Q, and from any two points N, Q in the curve, ſtraight lines N B, Q A parallel to the aſymptote C R be drawn to the other aſymptote C G, the figure A B N Q, bounded by the ſtraight lines N B, B A, A Q and the curve N Q, is called a *Hyperbolic Trapezium.*

Cor. A hyperbolic ſector C N Q and trapezium A B N Q upon the ſame curve are equal. For let C N cut A Q in P, and then, as by Cor. 2. Prop. XVII. the triangles C A Q, C B N are equal, the rectilineal trapezium A B N P is equal to the triangle C P Q. To theſe equals add the

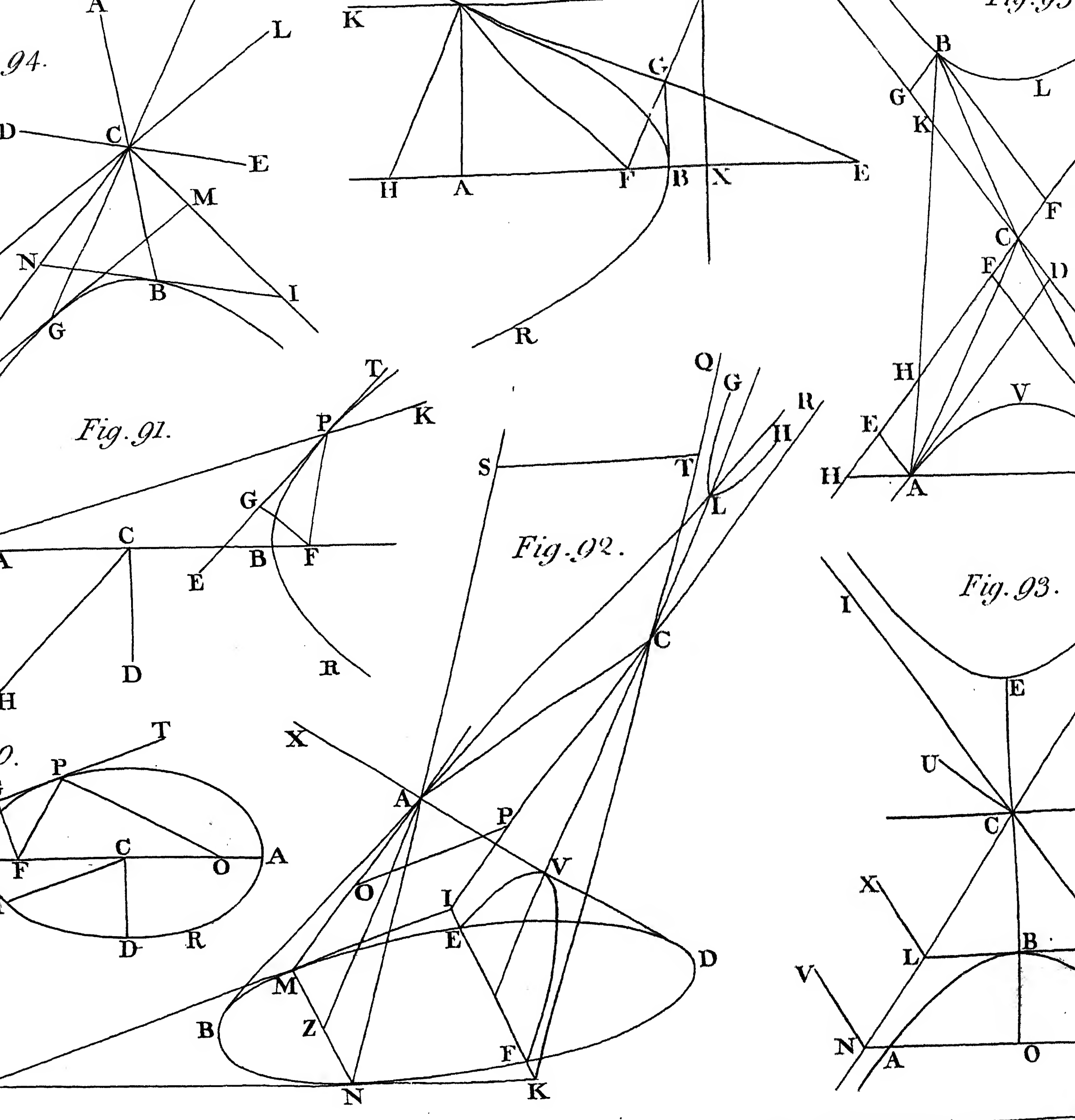


Fig. 94.
Fig. 91.
Fig. 92.
Fig. 93.

the figure P N Q, and then the hyperbolic ſector C N Q is equal to the hyperbolic trapezium A B N Q.

XVII.

Any ſegment as C A, intercepted between C the center and A a point in either aſymptote, is called an *Aſymptotic Segment*, and the point A is called its extremity.

XVIII.

Any ſtraight line in the plane of an hyperbola, or oppoſite hyperbolas, parallel to either of the aſymptotes is called an *Aſymptotic Secant*.

PROP. XVIII.

If from the points in which a ſtraight line cuts, and the point in which a ſtraight line parallel to it touches, an hyperbola, ſtraight lines parallel to one of the aſymptotes be drawn to the other, they will cut off from the center proportional aſymptotic ſegments: and, on the contrary, if from the extremities of three proportional aſymptotic ſegments ſtraight lines parallel to the other aſymptote be drawn to the curve of the hyperbola, the ſtraight line joining the extreme points in the curve will be parallel to the tangent paſſing through the middle point in the curve.

Let the ſtraight line M Q cut the hyperbola M N Q in the points M, Q, and let the ſtraight line G R, parallel to it, touch the hyperbola in N, and from Q, N, M let the ſtraight lines Q A, N B, M D, parallel to the aſymptote C R, be drawn to the aſymptote C G; the aſymptotic ſegments C A, C B, C D are proportionals. On the contrary, if C A, C B, C D be proportional aſymptotic ſegments, and ſtraight lines A Q, B N, D M, parallel to the aſymptote C R, be drawn to meet the curve of the hyperbola in Q, N, M, the ſtraight line Fig. 96.

BOOK III. M Q, joining the extreme points, is parallel to G R touching the hyperbola in the middle point N.

Part I. Let the ſecant M Q meet the aſymptotes C G, C R in the points H, K, and let the tangent G R, parallel to the ſecant, meet them in G, R. Then, as D M, A Q are parallel, (10. vi.) H M : H D : : K Q : C A, and therefore, by Cor. 1. Prop. XV. (and 14. v.) C A is equal to D H. Again, as by Cor. 1. Prop. XV. G N, N R are equal, and as B N, C R are parallel, (2. vi. and 14. v.) C B, B G are equal. Conſequently C A : C B : : D H : B G; and (4. vi.) D H : B G : : D M : B N, and therefore (11. v.) C A : C B : : D M : B N. But, by Cor. 2. Prop. XVII. D M : B N : : C B : C D; and therefore (11. v.) C A : C B : : C B : C D.

Part II. Upon the ſecond hypotheſis let M Q meet the aſymptotes in H, K, and let the tangent paſſing through N meet them in G, R. Then, as above, it may be demonſtrated that C A is equal to D H, and C B equal to B G; and therefore C A : C B : : D H : B G. But, by hypotheſis, C A : C B : : C B : C D; and, by Cor. 2. Prop. XVII. C B : C D : : D M : B N. Conſequently, (11. v.) D H : B G : : D M : B N, and by alteration D H : D M : : B G : B N; and therefore as D M, B N are parallel, the angles (6. vi.) D H M, B G N are equal, and (29. i.) H K, G R are parallel.

Cor. If two parallel ſtraight lines M Q, L F cut an hyperbola L M Q F, and from the points F, Q, M, L ſtraight lines F E, Q A, M D, L O parallel to the aſymptote C T be drawn to the other aſymptote C S, the aſymptotic ſegments C E, C A, C D, C O will be proportional. On the contrary, if the aſymptotic ſegments C E, C A, C D, C O be proportional, and E F, A Q, D M, L O, parallel to the aſymptote C T, be drawn to the curve, the ſtraight line L F joining the extreme points will be parallel to M Q joining the mean points. For, firſt,

first, if G R, parallel to M Q or L F, touch the hyperbola in N and N B, parallel to the asymptote C T, be drawn to the other asymptote C S, then by the above (and 17. vi.) C A × C D is equal to C B², and also C E × C O is equal to C B². Consequently C E × C O is equal to C A × C D, and C E : C A :: C D : C O.

Upon the second hypothesis let C B be a mean proportional between C A, C D, and let B N, parallel to C T, be drawn to the curve, and let G R touch the hyperbola in N. Then, by the second Part of the above, G R, M Q are parallel. Again, as by hypothesis C E : C A :: C D : C O, C E × C O is equal to C A × C D. But C B being a mean proportional between C A, C D, C B² is equal to C A × C D, and therefore C E × C O is equal to C B², and C B is a mean proportional between C E, C O. Consequently as before L F is parallel to G R, and therefore (30. i.) M Q, L F are parallel.

PROP. XIX.

If from the extremities of four proportional asymptotic segments asymptotic secants be drawn to the curve of the hyperbola, the hyperbolic trapezium between the first and second secant will be equal to the hyperbolic trapezium between the third and fourth. And if from the extremities of a series of asymptotic segments, in geometrical progression, asymptotic secants be drawn to the curve of the hyperbola, the hyperbolic trapezia between the first and second secant, the first and third, the first and fourth, and so on, will be in arithmetical progression.

Part I. Let M N Q be an hyperbola, of which C is the center and C T, C S the asymptotes, and in C S let C E be to C A as C D to C O, and let E F, A Q, D M, O L be asymptotic secants drawn to the curve, the hyperbolic Fig. 96.

BOOK III.

bolic trapezium E A Q F is equal to the hyperbolic trapezium D O L M.

For M Q, L F being drawn, they will be parallel, by Cor. Prop. XVIII. Draw C L, C M, C Q, C F. Let C V be the diameter to which the parallels M Q, L F are double ordinates, and let it meet M Q in T, and L F in V. Then (38. i.) the triangle C L V is equal to the triangle C F V, and the triangle C M T is equal to the triangle C Q T; and as the diameter C V bifects all ftraight lines parallel to M Q and terminated by the curve, the fpace T M L V is equal to the fpace T Q F V. Confequently (axiom 3. i.) the hyperbolic fector C F Q is equal to the hyperbolic fector C M L; and therefore, by Cor. to the fixteenth Definition, the hyperbolic trapezia E A Q F, D O L M are equal.

Part II. The reft remaining as above, let C A, C B, C D, C X, &c. be a feries of afymptotic fegments in geometrical progreffion, and let the afymptotic fecants A Q, B N, D M, X Y, &c. be drawn to the curve; the hyperbolic trapezia A B N Q, A D M Q, A X Y Q, &c. are in arithmetical progreffion.

For let G R touch the hyperbola in N, and then, by Prop. XVIII. it is parallel to M Q. Let the diameter C T pafs through N, and then, by Prop. II. it bifects M Q in T; and (38. i.) the triangles C T Q, C T M are equal. And as C T bifects every ftraight line parallel to M Q, and terminated by the curve, the fpace N M T is equal to the fpace N Q T. Confequently (axiom 3. i.) the hyperbolic fectors C Q N, C N M are equal; and therefore, by Cor. to the fixteenth Definition, the hyperbolic trapezia A B N Q, B D M N are equal. As, by hypothefis, C B is to C D as C D to C X, it may be proved, in the fame way, that the hyperbolic trapezia B D M N, D X Y M are equal; and the fame mode of proof may be extended to any number of terms. Confequently

ſequently the hyperbolic trapezia A B N Q, A D M Q, A X Y Q, &c. are in arithmetical progreſſion. BOOK III.

SCHOLIUM.

As the hyperbola and its aſymptotes may be indefinitely extended, it is evident that a ſeries of aſymptotic ſegments in geometrical progreſſion, and a correſponding ſeries of hyperbolic trapezia in arithmetical progreſſion, may be continued to any aſſigned number of terms. From the nature of logarithms, therefore, the ſeries of aſymptotic ſegments C A, C B, C D, &c. as above, is analogous to a ſeries of natural numbers in geometrical progreſſion, and the ſeries of hyperbolic trapezia A B N Q, A D M Q, &c. as above, is analogous to the logarithms of theſe natural numbers. To enter into an explanation of theſe analogies would be incompatible with the deſign of this work. The reader may find full information on the ſubject of logarithms in the volumes entitled, "Scriptores Logarithmici," publiſhed by Francis Maſeres, Eſq. F. R. S. Curſitor Baron of the Court of Exchequer. To this Gentleman the mathematical world are highly indebted for his perſevering exertions and liberality in the cauſe of ſcience. He has employed his great abilities in endeavours to render ſome of the moſt important ſubjects perſpicuous, and he has expended large ſums in the publication of ſcarce mathematical tracts, and made preſents of many copies of them, with the highly laudable motive of promoting learning and diſſeminating knowledge.

Within theſe twenty years, laſt paſt, much has been done in this country to facilitate the application of logarithms, and to extend their utility. In 1785 Dr. Hutton of Woolwich publiſhed in 8vo. extenſive Tables of them, to which he prefixed "A large and original Hiſtory of the Diſcoveries and Writings relating to

BOOK III. to thofe fubjects." Thefe Tables are judicioufly arranged, and are very valuable for general ufe. The hiftory prefixed to them is a mafterly performance; it gave rife to the publication mentioned above, entitled, "Scriptores Logarithmici."

In the year 1792 a quarto volume was publifhed, under the patronage of the Board of Longitude, entitled, "Tables of Logarithms of all Numbers, from 1 to 101000; and of the Sines and Tangents to every fecond of the Quadrant. By Michael Taylor, Author of the Sexagefimal Table." As Mr. Taylor died before the Tables were entirely printed, the Rev. Dr. Mafkelyne, Aftronomer Royal, fuperintended their completion. He alfo wrote the Preface and Precepts for the ufe of the Tables; and thefe he executed with that care, learning, and ability, for which he is fo juftly celebrated in every part of the world where either the theory, or practical utility, of Aftronomy is underftood. As Taylor's Tables are accurate, and more extenfive than any other extant, the volume is an excellent refource to thofe who aim at a fuperior degree of precifion in their calculations.

In a fmall quarto volume of mathematical Effays, publifhed in 1788 by the Rev. John Hellins, (now Vicar of Potters' Pury, Northamptonfhire, and F.R.S.) there are two Effays on Logarithms, which difplay an intimate knowledge of the fubject. A fcientific reader will find much gratification in the perufal of thefe Effays.

DEFINITION.

XIX.

Fig. 97. If A B be a tranfverfe diameter of the oppofite hyperbolas A, B, and D E the fecond diameter conjugate to it, and if D E be a tranfverfe diameter of the oppofite

ſite hyperbolas D, E, and A B the ſecond diameter conjugate to it; the hyperbolas D, E are called the *Conjugate Hyperbolas* to one or both of the oppoſite hyperbolas A, B, and, on the contrary, the hyperbolas A, B are called the *Conjugate Hyperbolas* to one or both of the oppoſite hyperbolas D, E. When all the four hyperbolas A, D, B, E are mentioned together, they are called *Conjugate Hyperbolas*.

Cor. If the diameters A B, D E cut one another in C, it is evident that C is the common center of the conjugate hyperbolas.

PROP. XX.

One of the aſymptotes of an hyperbola is parallel to, and the other biſects, a ſtraight line joining the vertices of any two conjugate diameters: and the vertices of ſecond diameters of an hyperbola are in the curves of the hyperbolas conjugate to it.

Part I. Let A B, D E be any two conjugate diameters of the hyperbola A H, and let C F, C G be its aſymptotes, C being the center; one of the aſymptotes, as C F, is parallel to A E the ſtraight line joining the vertices A, E, and the other aſymptote C G biſects A E. Fig. 97.

For let F G touch the hyperbola H A in A and meet the aſymptotes in F and G. Then, by Cor. 3. and 1. Prop. XV. F G is equal and parallel to D E, and C E is equal to F A and alſo to A G. Conſequently (33. i.) the aſymptote C F is parallel to A E. Alſo the angle (29. i.) C E L is equal to the angle L A G, and the angle E C L to the angle A G L, and therefore (26. i.) A L is equal to E L, and A E is biſected by the aſymptote C G.

Part II. Let K H be any tranſverſe diameter of the oppoſite hyperbolas A H, B K, and let M N be the ſecond

BOOK III. cond diameter conjugate to it; the vertices M, N of this ſecond diameter are in the curves of the hyperbolas conjugate to the hyperbolas A H, B K.

For let D, E be the hyperbolas conjugate to A H, B K, and let A B, D E be the conjugate diameters common to the conjugate hyperbolas, as in the nineteenth Definition, and C the center. Let F C, G C be the aſymptotes of the oppoſite hyperbolas A, B. Draw B E, K N, and let them meet the aſymptote F C in P and Q; and then, by Part I. B E, K N are parallel to the aſymptote G C, and they are biſected by the aſymptote F C in P and Q. Conſequently, by Cor. 2. Prop. XVII. C P : C Q : : K Q : B P; and therefore, on account of the equals, C P : C Q : : Q N : P E, and as E is in the curve of the hyperbola E, N muſt be in the curve of the ſame hyperbola E, by Cor. 3. Prop. XVII.

Cor. A ſtraight line parallel to one of the aſymptotes, and terminated by the curves of the conjugate hyperbolas, is biſected by the other aſymptote.

PROP. XXI.

A quadrilateral figure, whoſe ſides paſs through the vertices of any two conjugate diameters of an ellipſe, or conjugate hyperbolas, and touch the ellipſe or hyperbolas, is a parallelogram, and is equal to the rectangle under the axes of the ellipſe or hyperbolas.

Fig. 98. 99. Let M S L R be a quadrilateral figure, whoſe ſides M S, S L, L R, R M paſs through F, G, H, K, the vertices of the conjugate diameters F H, G K of the ellipſe F G H K, or the conjugate hyperbolas F, G, H, K, and in theſe points touch the curve or curves; the figure M S L R is a parallelogram, and is equal to the rectangle under the axes A B, D E of the ellipſe or hyperbolas,

For,

For, by Cor. 2. Prop. IV. Book II. the tangents M S, R L are parallel to the diameter K G, and the tangents M R, L S are parallel to the diameter F H. The quadrilateral figure M S L R is therefore a parallelogram. BOOK III.

Again, let C be the center, and let C I be perpendicular to the tangent S L. Then as F H is bisected in C, the parallelogram G H is a fourth part of the parallelogram L M; and, by Cor. 1. Prop. XIX. Book II. C I : C B :: C D : C H, and C I × C H is equal to C B × C D. But (35. i.) C I × C H is equal to the parallelogram G H, and C B × C D is a fourth part of the rectangle under the axes A B, D E. Consequently the parallelogram L M is equal to the rectangle under the axes A B, D E.

Cor. All parallelograms contained under tangents, passing through vertices of conjugate diameters of an ellipse, or conjugate hyperbolas, are equal to one another, and each of them is equal to the rectangle under the axes.

The twelfth Lemma of Sir Isaac Newton's Principia, Lib. I. and the tenth Proposition, depending upon it, are evident from the above.

PROP. XXII.

If through a point in the curve of an hyperbola two straight lines be drawn parallel to the asymptotes and meeting a diameter, the semidiameter will be a mean proportional between the segments of the diameter between the center and the points of concourse.

Through the point I in the curve of the hyperbola I B let the straight lines I T, I A be drawn parallel to the asymptotes C E, C G, and first let them meet the transverse diameter D B in the points T, A, and let C be the Fig. 100.

BOOK III. the center; the femidiameter C B is a mean proportional between C T, C A.

For let I A meet the afymptote C E in F, and let B H, T K be parallel to I F, or to the afymptote C G, and let them meet C E in H and K. Then (34. i.) K T, F I are equal; and (4. vi.) C K : C H :: K T or its equal F I : H B. But, by Cor. 2. Prop. XVII. F I : H B :: C H : C F, and therefore (11. v.) C K : C H :: C H : C F. Confequently, on account of the parallels K T, H B, F A, C T : C B :: C B : C A.

Secondly, the reft remaining as above, let the ftraight lines I T, I A meet the fecond diameter L M in the points N, O, and let I A meet the curve of the hyperbola P L, conjugate to B I, in P; and let P Q be parallel to C E or I N, and meet the diameter L M in Q. Then, by the above, C Q : C L :: C L : C O. But, by Cor. Prop. XX. I F, F P are equal, and therefore, as P Q, I N are parallel, C Q is equal to C N. Confequently C N : C L :: C L : C O.

Fig. 101. 102. *Cor.* 1. If two ftraight lines T N, T P, touching an hyperbola, or oppofite hyperbolas, in N and P, meet one another in T, and if a ftraight line T I parallel to one of the afymptotes meet the curve in I, a ftraight line I A parallel to the other afymptote and meeting N P, the ftraight line joining the points of contact, in A will bifect N P in A. This is evident from the above, and Prop. VII. Book II.

Cor. 2. The reft remaining as in the preceding Cor. if T I produced meet N P in E, T E will be bifected in I. For let T L parallel to I A, or an afymptote, meet the curve in L, and then, by Cor. 1. L A parallel to T I will meet N P in A, the point in which N P is bifected; and T I A L is a parallelogram. Draw I L, and let it meet the diameter C B A in G. Then (34. i.) I A, T L are equal, and (29. i.) the triangles A I G, T L G are equi-

sc.

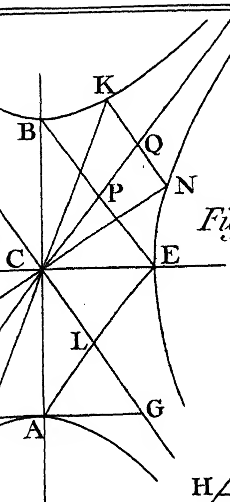


Fig. 97.

Fig. 96.

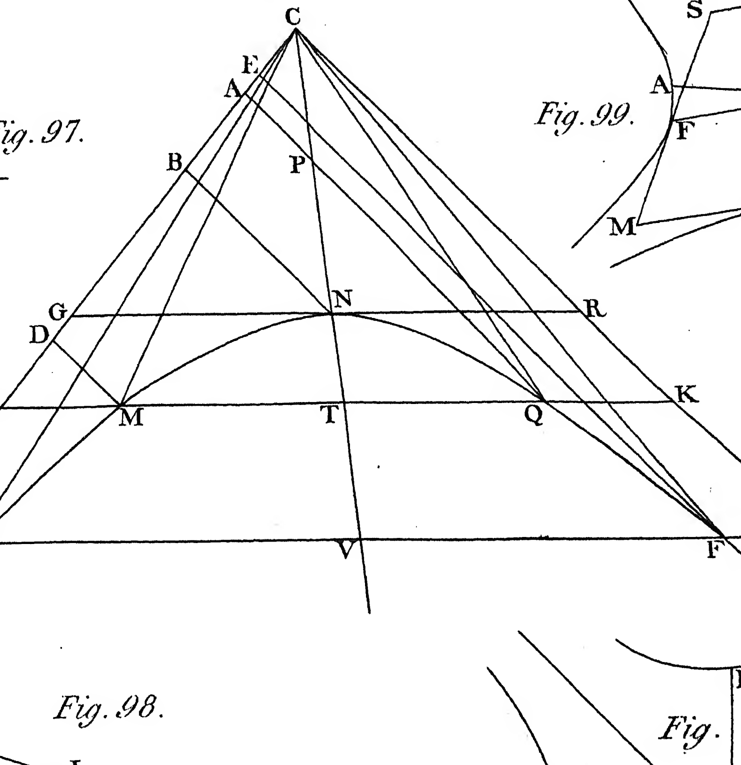


Fig. 99.

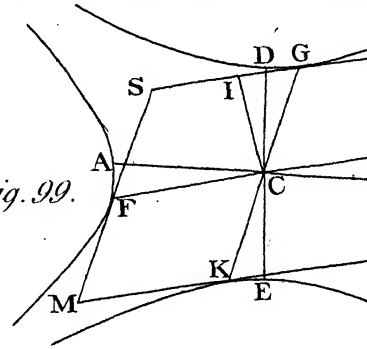


Fig. 98.

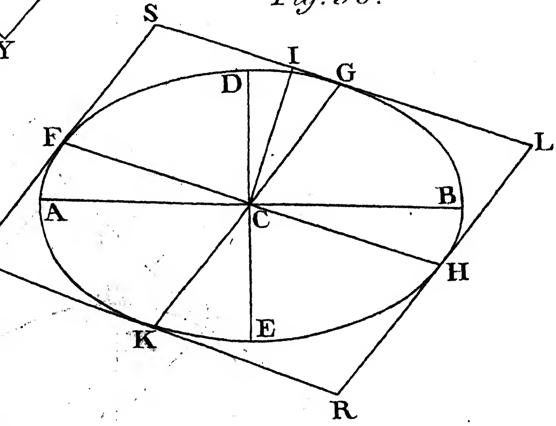
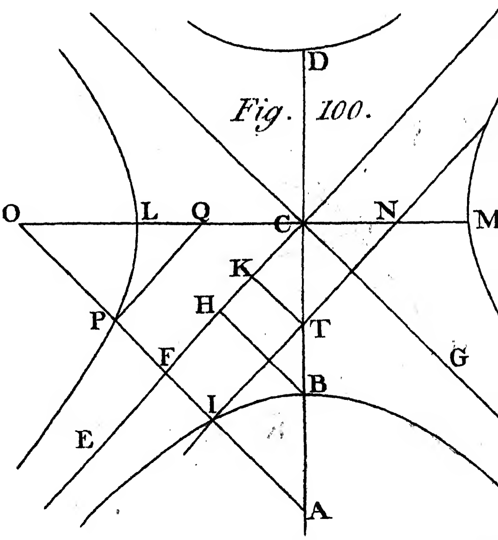


Fig. 100.



equiangular; and therefore (26. i.) I G, G L are equal, and T G is equal to G A. The ſtraight lines I L, N P are therefore ordinates to the diameter C B A, and conſequently, by Prop. II. they are parallel; and (2. vi.) T G : G A :: T I : I E. The ſtraight line T E is therefore biſected in I. BOOK III.

PROP. XXIII.

If a diameter of a parabola, or a ſtraight line parallel to an aſymptote of an hyperbola, meet two tangents and the ſtraight lines joining the points of contact, the ſquare of its ſegment between the curve and the ſtraight line joining the points of contact will be equal to the rectangle under the ſegments between the curve and tangents.

If the diameter of the parabola, or the ſtraight line parallel to an aſymptote of an hyperbola, paſs through the point in which the tangents meet one another, the Propoſition is evident from Prop. V. and Cor. 2. Prop. XXII. but if not, let the two ſtraight lines M N, M O touch the parabola, hyperbola, or oppoſite hyperbolas, in the points N, O, and let T X a diameter of the parabola, or a ſtraight line parallel to an aſymptote, meet the parabola or either hyperbola in E, the tangents in A, D, and the ſtraight line joining the points of contact in B; the ſquare of E B is equal to the rectangle under A E, E D. Fig. 103. 104. 105.

Caſe 1. If the ſtraight lines touching the parabola or hyperbola meet one another in M, from the point D in which T X meets one of the tangents draw D L parallel to the other tangent M O, and let it meet the curve in P, L, and the ſtraight line N O in K. Then, by Prop. XVII. Book I. the ſquare of D K is equal to the rectangle under L D, D P; and (4. and 22. vi.) Fig. 103. 104.

A B²

BOOK III. $AB^2 : DB^2 :: AO^2 : DK^2$ or $LD \times DP$. But as TX is parallel to a ſide of the cone in which the ſection was formed, by Prop. XVI. Book I. $AO^2 : LD \times DP ::$ $AE : DE$; and therefore (11. v.) $AE : DE :: AB^2 :$ DB^2. Conſequently, by Lemma VIII. $AE : BE ::$ $BE : DE$; and (17. vi.) $AE \times ED$ is equal to BE^2.

Fig. 105. Caſe 2. If the ſtraight lines touching the oppoſite hyperbolas meet one another in M, from A or D, ſuppoſe D, draw the ſtraight line DK parallel to the tangent AO, and let it meet NO in K. Through the points A, D draw the ſtraight lines GH, PL parallel to one another, and let them meet the oppoſite hyperbolas in the points G, H and P, L. Then, by Prop. V. Book II. $GA \times AH$ is to AO^2 as the ſquare of the ſemidiameter parallel to GH to the ſquare of the ſemidiameter parallel to AO. And, by Cor. 1. Prop. XVII. Book I. and Prop. V. Book II. $PD \times DL$ is to DK^2 as the ſquares of the ſame ſemidiameters. Conſequently (11. v. and alternation) $GA \times AH : PD \times DL :: AO^2 :$ DK^2. But (4. and 22. vi.) $AO^2 : DK^2 :: AB^2 : DB^2$; and, by Prop. XVI. Book I. $GA \times AH : PD \times DL ::$ $AE : DE$. Conſequently $AE : DE :: AB^2 : DB^2$, and therefore, by Lemma VIII. $AE : BE :: BE : DE$, and $AE \times DE$ is equal to BE^2.

Fig. 105. Caſe 3. If the ſtraight lines AO, DN touching the oppoſite hyperbolas be parallel, then the triangles ABO, DBN will be equiangular, and the proportion will be $AO^2 : DN^2 :: AB^2 : DB^2$; and in this caſe, by Prop. XVI. Book I. $AE : DE :: AO^2 : DN^2$. Conſequently $AE : DE :: AB^2 : DB^2$, and, by Lemma VIII. $AE :$ $BE :: BE : DE$, and $AE \times DE$ is equal to BE^2.

PROP. XXIV.

If from two given points in the curve of a parabola, or hyperbola, or the curves of oppoſite hyperbolas, two ſtraight

ſtraight lines be inflected to any third point in the curve of the parabola, or in the curve of either of the oppoſite hyperbolas, and if they meet a diameter of the parabola, or a ſtraight line parallel to an aſymptote of the hyperbola; the ſegments of this laſt mentioned line, between the inflected lines and the point in which it meets the curve of the ſection, will be to one another in the ſame ratio, wherever the point may be in the curve to which the lines are inflected. BOOK III.

Let N, M be two given points in the curve of the parabola, hyperbola, or oppoſite hyperbolas, and let the ſtraight lines N O, M O inflected from them to any point O in the curve of the parabola, or in the curve of either hyperbola, meet in B, C the ſtraight line T X, a diameter of the parabola, or parallel to an aſymptote of the hyperbola, and let T X meet the curve in E; the ſegments E B, E C are to one another in the ſame ratio, wherever the point O may be taken in the curve. Fig. 106. 107. 108.

For let tangents paſſing through M, N, O meet T X in F, D, A. Draw M N, and let it meet T X in G; and by Prop. XXIII. $ED : EG :: EG : EF$. Again, by Prop. XXIII. EB^2 is equal to $AE \times ED$, and EC^2 is equal to $AE \times EF$. Conſequently $EB^2 : EC^2 :: AE \times ED : AE \times EF ::$ (1. vi.) $ED : EF$. But, by the above, (and Cor. 2. 20. vi.) $ED : EF :: ED^2 : EG^2$; and therefore (11. v.) $EB^2 : EC^2 :: ED^2 : EG^2$, and (22. vi.) $EB : EC :: ED : EG$. But as the points M, N are given, or fixed, and as the ſtraight line T X is given in poſition, the ſegments E D, E G remain the ſame. Conſequently, if the point O be moved round the curve of the parabola, hyperbola, or oppoſite hyperbola, the ſegments E B, E C of the ſtraight line T X, between the inflected lines and the curve, will be to one another in the ſame ratio, wherever the point O may be in the curve.

 SCHO-

BOOK III.

SCHOLIUM.

It may be proper in this place to direct the attention of the reader to methods of ascertaining certain particulars in a conic section, supposing the curves of the sections to be given, or straight lines to be given, for the description of the curves. These methods, now to be described, might have been delivered as Corollaries to the Propositions on which they depend, or they might have been put into the form of Problems; but it appeared more adviseable to reserve them for a series of articles in a Scholium. A cautious desire against interrupting the reader in the acquisition of new truths suggested this delay. The ease with which they are deduced from the preceding Propositions, and the importance of the articles themselves, induced the author to think that the following was the most proper manner of delivering them, and the most proper place to insert them.

Fig. 109. 110. 1. Let the ellipse A D B E, or the opposite hyperbolas A, D B E, be given to find the center.

In the ellipse and in either hyperbola draw the two parallel straight lines D E, F G, and draw A B bisecting D E in H and F G in K. Let A B meet the curve of the ellipse, or the curves of the opposite hyperbolas, in A and B. Bisect A B in C, and C will be the center, by Cor. 1. Prop. III. Book II.

If only one hyperbola D B E be given, two other straight lines must be drawn parallel to one another, but not parallel to D E, F G, and a straight line being drawn bisecting them will be a diameter. Its concourse therefore with A B will determine the center.

2. The curve of a conic section and a point in it being given, let be required to draw a diameter through the given point.

If the section be an ellipse, or hyperbola, find the cen-

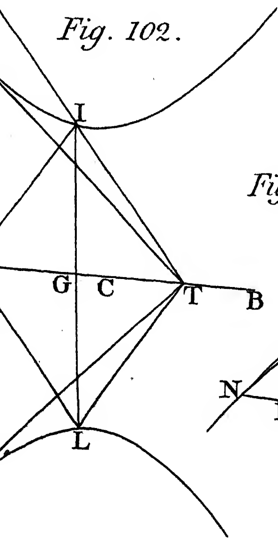


Fig. 102.

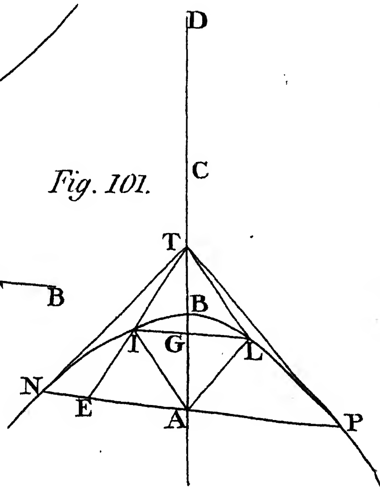


Fig. 101.

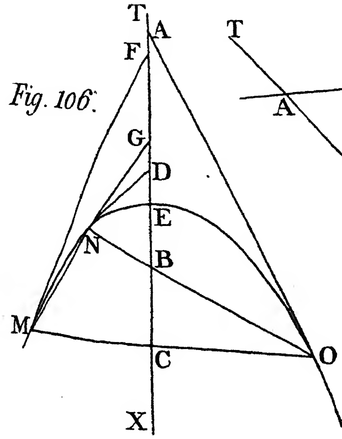


Fig. 106.

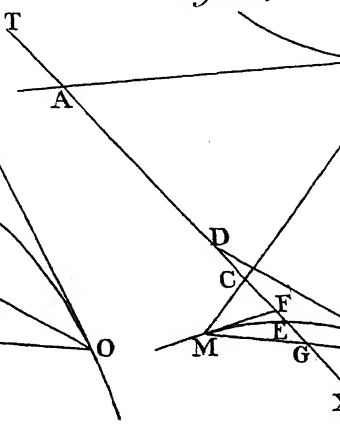


Fig. 107.

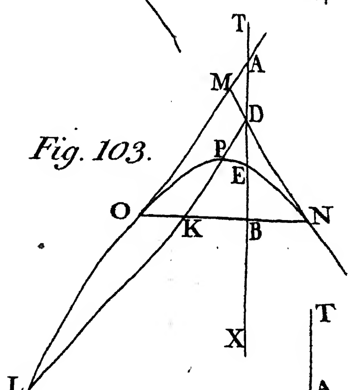


Fig. 103.

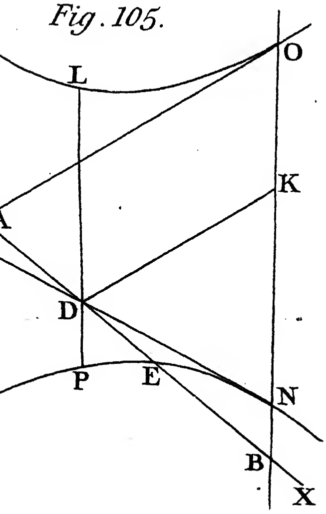


Fig. 105.

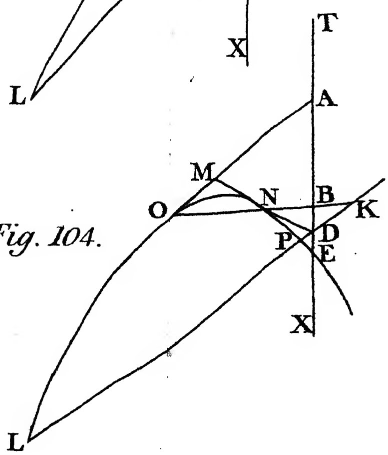


Fig. 104.

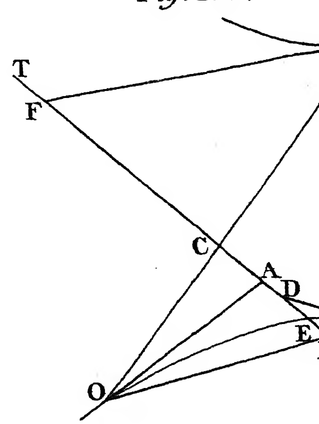


Fig. 108.

center by the preceding article, and through the center and the given point draw a diameter. If the section be a parabola, find a diameter, by Cor. 3. Prop. II. of this Book, and parallel to it draw a ſtraight line through the given point; and, by Cor. 1. to the firſt Definition, this will be the diameter required.

3. The curve and a diameter A B of a conic ſection being given, and any point G in the curve beſides a vertex of the given diameter, let it be required to draw a ſtraight line from G ordinately applied to the diameter.

First, let the ſection A G B D be an ellipſe. Find the center C by the firſt article, and through it draw the diameter G L. Through the vertex L draw the ſtraight line L F parallel to A B. Then if L F touch the ellipſe, A B, L G will be conjugate diameters, by Cor. 2. Prop. IV. Book II. and L G will be ordinately applied to A B. But if L F do not touch the ellipſe, let it meet the curve again in F. Draw G F, and let it meet A B in K, and G F will be the ordinate required. For, as C K, L F are parallel, (2. vi.) G C : G K :: C L : K F, and therefore (14. v.) G F is biſected in K. Secondly, let the ſection F B G be an hyperbola, or parabola. In the diameter A B take any point M, and the ſtraight line G M being drawn, produce it to L, ſo that M L may be equal to G M. Then if the point L be in the curve, G L will be the ordinate required; but if L be not in the curve, draw the ſtraight line L F parallel to A B, and let it meet the curve in F, according to Prop. IX. Book I. or Cor. 1. Def. I. of this. Draw G F, and it will be the ordinate required. For, if it meet A B in K, it may be proved, as above, that G F is biſected in K. Fig. 109. Fig. 110.

4. The curve of a conic ſection F B G being given, and a point B being given in it, let it be required to draw a ſtraight line through B to touch the ſection. Fig. 110.

Through

BOOK III.

Through B draw A B, a diameter of the section, and, by the last article, draw G F an ordinate to it. Through the vertex B draw the straight line T B parallel to G F, and, by Prop. II. T B will be the tangent required.

Fig. 111. 5. Two unequal straight lines A B, D E being given, bisecting one another in C at right angles, let it be required to describe the curve of an ellipse, of which A B, D E shall be the axes, and C the center.

Let A B be greater than D E, and consequently the transverse axis. Find the foci F, O in A B, by Cor. 2. Def. XI. Book II. Let the ends of a thread or string, equal in length to A B, be fixed to the points F, O. By means of the pin P let the thread or string be stretched; and, while it continues uniformly tense, let the end or point of the pin P move round in the plane, in which A B, D E are situated, till it return to the same place from which it began to move. The line traced by the end or point of the pin P is the curve of an ellipse, as is evident from Prop. XIII. Book II. and A B, D E are the axes.

Fig. 112. 6. Two straight lines A B, D E being given, bisecting one another in C at right angles, let it be required to describe the curve of an hyperbola, of which A B shall be the transverse and D E the conjugate axes.

In A B produced both ways let the foci F, O be found, by Cor. 3. Def. XI. Book II. Let one end of a thread or string F P R be fixed to the point F, and let the other end be fixed to the extremity R of the ruler O R, and let the length of the ruler exceed the length of the thread or string by the straight line A B. Let O, the other extremity of the ruler, be fixed to the point O, and let the ruler revolve about O as a center. By means of the pin P let the thread or string be stretched, and let the part between P and R be kept close to the edge of the ruler; and while the ruler revolves,

let the end or point of the pin P trace the line G B H in the plane in which A B, D E are situated. The line G B H will be the curve of an hyperbola, of which A B is the transverse and D E is the conjugate axes, and C is the center, as is evident from Prop. XIII. Book II.

7. The straight line D X, of indefinite length, being given, and F being a point given without it, let it be required to describe the curve of a parabola, of which D X shall be the directrix and F the focus. Fig. 113.

Place the edge of a ruler R D X L along the line D X, and keep it fixed in that position. Let G Y E be a ruler of such a form that the part G Y may slide along the edge D X of the fixed ruler R D X L, and the part Y E may be always perpendicular to D X. Let E P F be a thread or string of the same length with the part Y E of the moving ruler, and let one end of it be fixed to the ruler at E, and let the other end be fixed to the point F. By means of the pin P let the thread or string be stretched, and the part between P and E be kept close to the edge of the ruler. While the ruler G Y E slides along the edge D X of the fixed ruler, and the thread or string is kept uniformly tense, let the end or point of the pin P trace the line A P B C on the plane, in which the line D X and the point F are situated. The line A P B C will be the curve of a parabola, of which D X is the directrix and F the focus, as is evident from Cor. 3. Prop. VIII.

Several writers on Conic Sections have defined the ellipse, hyperbola, and parabola by the description in Article 5, 6, and 7 respectively; and from these descriptions, as founded on a primary, they have deduced other properties of the sections.

8. Two straight lines A B, D E being given, bisecting one another in C but not at right angles, let it be re- Fig. 114. 115.

BOOK III.

required to deſcribe an ellipſe, or hyperbola, of which A B, D E ſhall be conjugate diameters, and C the center.

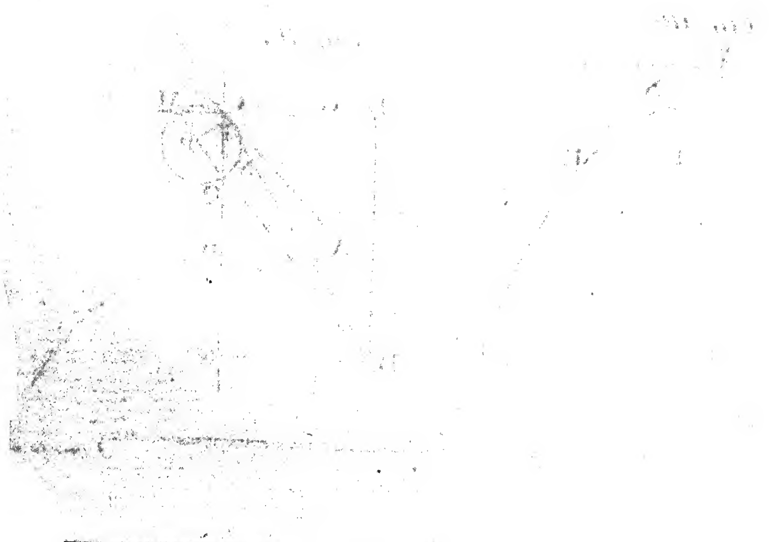
In C D, produced in the ellipſe but between C and D in the hyperbola, take the point N, ſo that the rectangle under C D, D N may be equal to the ſquare of C B. Through D draw the ſtraight line M Q parallel to A B, and biſect C N in I. Draw I P perpendicular to C N, and let it meet M Q in P; and then it is evident (4. i.) that ſtraight lines drawn from P to N and C will be equal. With P as a center therefore, and P N or P C as a diſtance, let the circle M C Q N be deſcribed, and let it meet the ſtraight line M Q in M and Q. Draw the ſtraight lines Q C, M C; and from D draw D K perpendicular to Q C, and D H perpendicular to M C. In Q C take C L, C R each a mean proportional between C Q, C K; and in M C take C F and C G each a mean proportional between C M, C H. Then will R L, G F be the axes of the ellipſe, or hyperbola, propoſed to be deſcribed, as is evident from Cor. 2. Prop. IX. (and 31. iii.) and Prop. VII. Book II. Conſequently the foci may be found, and the deſcriptions of the curves may be completed, as in the 5th and 6th Articles.

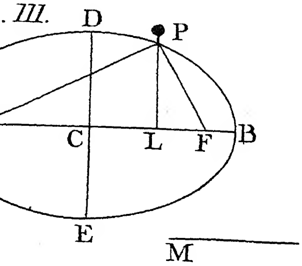
Fig. 116. 9. The ſtraight line G E being given in poſition and magnitude, and the ſtraight line A B biſecting it in B, let it be required to deſcribe a parabola, of which A B ſhall be a diameter, and G E a double ordinate to it. Let the ſtraight line P be a third proportional to A B, B G, and produce B A to Y, ſo that A Y may be a fourth part of P. Through Y draw D X at right angles to Y B, and through A draw A N parallel to G E. Make the angle N A F equal to the angle N A Y, and make A F equal to A Y. A parabola deſcribed with the focus F and the directrix D X, as in the 7th Article, will be the ſection required, as is evident from Prop. II. IX. and Cor. 2. and 3. Prop. XI.

10. The

and P L an ordinate to it being given, let it be required to find the diameter conjugate to A B. Fig. 111. 112.

Let the straight line M be a mean proportional (13. vi.) between the abscisses A F, F B; and, C being the center of the section, let M be to P L as C B to C D a straight line parallel to P L. Then will C D be half the conjugate diameter required. For, by hypothesis, (and 22. vi.) $M^2 : PL^2 :: CB^2 : CD^2$. But (17. vi.) $M^2 = AL \times LB$, and therefore $AL \times LB : PL^2 :: CB^2 : CD^2$. Consequently C D is the semiconjugate diameter to A B, by Prop. IV. and V. and Def. VIII. Book II.





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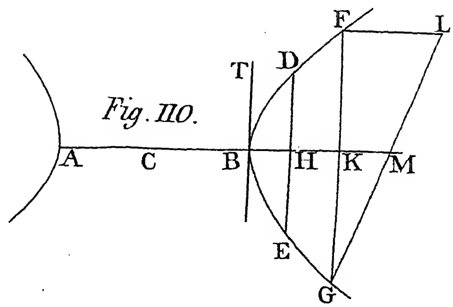

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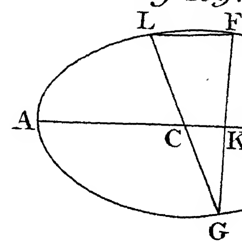

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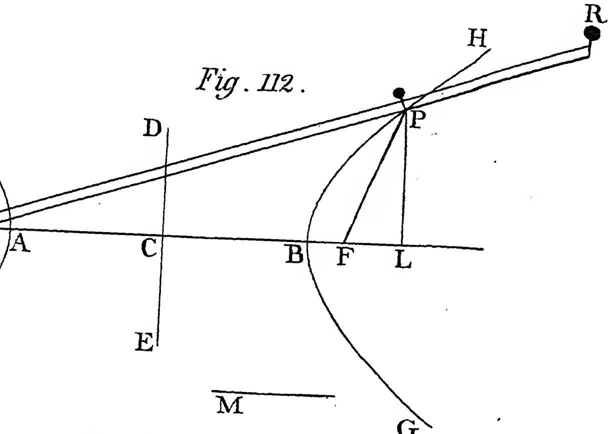

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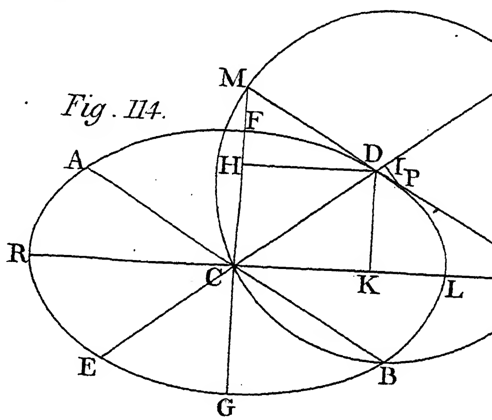


Fig. 114.

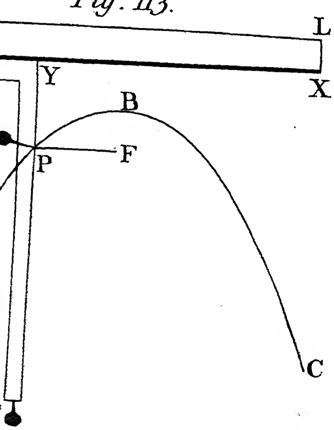
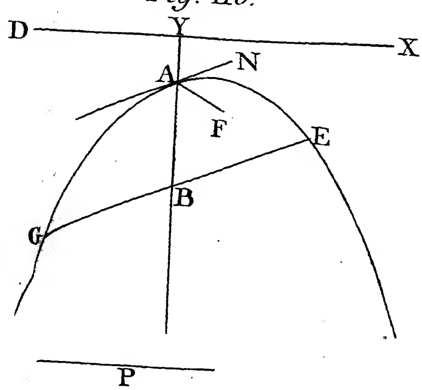


Fig. 113.



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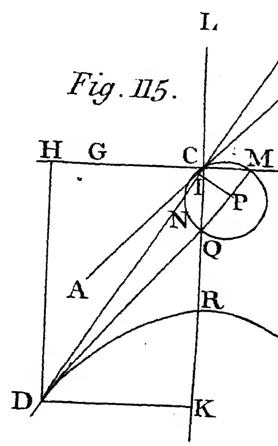

Fig. 115.

LEMMAS

FOR

THE FOURTH BOOK

OF

CONIC SECTIONS.

LEMMA IX.*

Let it be required to draw a straight line to touch two given circles A L B, D E M.

Let C be the center of A L B, and F the center of D E M, and draw C F. Let C G be the excess of the radius of A L B above the radius of D E M; and with C as a center, and C G as a distance, describe the circle H G. From the point F draw F H (17. iii.) touching the circle H G in H. Draw C H, and let it meet the circumference of A L B in A; and draw F D parallel to C A, and let it meet the circumference of D E M in D. Draw A D, and it will touch the given circles. For by the construction A H, D F are equal and parallel, and (16. iii.) A H F is a right angle. Consequently (33. i.) A D F H is a parallelogram, and (Cor. 46. i.) Fig. 128.

* The numbering of the Lemmas is continued from those prefixed to the first Book. This manner of numbering them was found most convenient for reference.

the

the angles at A and D are right ones, and therefore (16. iii.) A D touches the circles.

If it be required to draw a ſtraight line to touch the circles and cut C F, let C F be ſo divided in K that C K may be to K F as the radius of the circle A L B to the radius of the circle D E M. Draw K L (17. iii.) to touch the circle A L B in L. Draw C L and F M parallel to it, and let F M meet L K in M; and L K will touch the circle D E M in M. For (15. and 29. i.) the triangles C K L, F K M are equiangular, and therefore (4. vi.) C K : K F : : C L : F M. Conſequently, by the conſtruction, the point M is in the circumference of the circle D E M, and as the angles F M K, C L K are equal, and the angle C L K a right one, L M (16. iii.) touches the circle D E M in M.

LEMMA X.

If a magnitude A *be to a magnitude* B *as a magnitude* C *to a magnitude* D; *then* A *will be to* B *as the difference of the antecedents* A, C *to the difference of the conſequents* B, D.

For let A be greater than C, and conſequently (14. v.) B greater than D. Then, by alternation, A : C : : B : D, and (17. v.) A — C : C : : B — D : D; and again, by alternation, C : D : : A — C : B — D. But, by hypotheſis, A : B : : C : D, and therefore (11. v.) A : B : : A — C : B — D.

DEFINITIONS.

I.

Fig. 129. 130. If the ſtraight line A D be ſo divided in the points B, C, that the whole line A D be to one of the extreme parts A B, as the other extreme part D C to the middle part B C, the ſtraight line A D is ſaid to be *Harmonically*

monically divided; and the points A, B, C, D are called the *Points of harmonical division*, or *Harmonical Points*.

Cor. 1. It is evident that each extreme part is greater than the middle part.

Cor. 2. If the two extreme points A, D, and B one of the middle points of an harmonical divifion be given, the other point C may be found (10. vi.) by dividing the fegment B D in C, fo that the part C D may be to B C as A D to A B. It is evident that no other point befides C can be found, which can be a fourth point of this divifion.

Cor. 3. The two middle points B, C, and A one of the extreme points of an harmonical divifion being given, the other point D may be found. For from the point A draw the ftraight line A E, and as A B to B C, fo let A E be to the fegment E F, taken towards A. Draw F C, and let E D drawn parallel to F C meet A C produced in D. Then (2. vi.) A D : D C : : A E : E F; and therefore by the conftruction A D : D C : : A B : B C, and confequently D is the other point of the divifion. It is evident that no other point befides D can be found, which can be a fourth point of this divifion. For, by converfion, as the excefs of A B above B C to B C, fo is A C to C D. Fig. 130.

II.

If the ftraight line *a d* be divided harmonically in the points *a*, *b*, *c*, *d*, and four ftraight lines *a* E, *b* E, *c* E, *d* E, any way produced through the points of divifion, be parallel or meet one another in the point E; thefe four ftraight lines are called *Harmonicals*. Fig. 129. 130.

Cor. Every thing remaining as above, any ftraight line A D, parallel to *a d*, and meeting the harmonicals in A, B, C, D, will be harmonically divided in thefe points. For, (29. and 15. i.) on account of the equiangular triangles,

angles, $ad : dE :: AD : DE$, and $dE : dc :: DE : DC$. Consequently,

$$ad : dE : dc$$
$$AD : DE : DC,$$

and (22. v.) $ad : dc :: AD : DC$. Again $ab : bE :: AB : BE$, and $bE : bc :: BE : BC$; and therefore

$$ab : bE : bc$$
$$AB : BE : BC.$$

Consequently (22. v.) $ab : bc :: AB : BC$; and therefore (11. v.) $AD : DC :: AB : BC$, as, by hypothesis, $ad : dc :: ab : bc$.

LEMMA XI.

Fig. 129. 130. *The rest remaining as in the first and second Definitions and their Corollaries, if a straight line* GH *parallel to any one of the harmonicals* aE, bE, cE, dE, *meet the other three, it will be bisected in the middle point of concourse. And, on the contrary, if four straight lines* ED, EC, EB, EA *meet one another in* E, *and if the straight line* GH *parallel to any one of them, and meeting the other three, be bisected in the middle point of concourse, the straight lines* ED, EC, EB, EA *will be harmonicals.*

Fig. 130. Part I. First, let the straight line GH be parallel to the harmonical dE, and meet aE, bE, cE in G, B, H; GH is bisected in the middle point B. For through B draw the straight line AD parallel to ad, and meeting the harmonicals aE, cE, dE in A, C, D. Then the straight line AD is harmonically divided, by the Cor. to the second Definition, and therefore $DA : AB :: DC : CB$. But (4. vi.) $DA : AB :: DE : BG$; and $DC : CB :: DE : BH$. Consequently (11. v.) $DE : BG :: DE : BH$, and therefore (14. v.) GB, BH are equal.

Se-

Secondly, let the ſtraight line G H be parallel to the 'armonical c E, and meet a E, b E, d E in A, H, G; it ᵣill be biſected in the middle point A. For through draw the ſtraight line A D parallel to $a\,d$, and meet- ıg the harmonicals b E, c E, d E in B, C, D. Then, y the Cor. to the ſecond Definition, A D : D C : : A B : C. But (4. vi.) A D : D C : : G A : E C; and A B : C : : A H : E C. Conſequently (11. v.) G A : E C : : H : E C, and therefore (14. v.) G A, A H are equal. Fig. 129.

Part II. Firſt, let the ſtraight line G H be parallel to D, and meet the ſtraight lines E C, E B, E A in H, B, , and let it be biſected in the middle point B. Through draw the ſtraight line A D, and let it meet the ſtraight ıes E A, E C, E D in A, C, D. Then (4. vi.) D E : B G : : A : A B, and D E : B H : : D C : C B. Conſequently . and 11. v.) D A : A B : : D C : C B. Fig. 130.

Secondly, let the ſtraight line G H be parallel to E C, ıd meet the ſtraight lines E D, E B, E A in the points H, A, and let it be biſected in the middle point A. ırough A draw the ſtraight line A D, meeting the aight lines E B, E C, E D in B, C, D. Then (4. vi.) A : E C : : A D : D C; and A H : E C : : A B : B C. ›nſequently (7. and 11. v.) A D : D C : : A B : B C. Fig. 129.

LEMMA XII.

four harmonicals meet any ſtraight line, the ſtraight line will be harmonically divided in the points of concourſe.

If the harmonicals are parallel to one another, this is dent (from 10. vi.), but if not, let a E, b E, c E, d E four harmonicals, and let them meet any ſtraight e A D in A, B, C, D. Through B draw the ſtraight e G H parallel to d D, and let it meet the ſtraight e E A in G, and E C in H. Then, by the preceding Fig. 130.

Lemma,

Lemma, G H is biſected in B; and (4. vi.) D A : A B : : D E : B G or B H; and D E : B H : : D C : C B. Conſequently (11. v.) D A : A B : : D C : C B.

A GEO-

A

GEOMETRICAL TREATISE

OF

CONIC SECTIONS.

BOOK IV.

Of ſimilar Sections, general Properties, Circles having the ſame curvature with the Sections in given points, and of ſtraight lines cut harmonically by the Sections. This Book alſo contains Problems uſeful in the Theory of Aſtronomy, and Methods of finding two mean proportionals and of triſecting an angle, by means of the Sections.

DEFINITIONS.

I.

TWO ſegments of conic ſections are called *Similar Segments*, if a rectilineal figure can be inſcribed in one of them ſimilar to any rectilineal figure inſcribed in the other.

II.

Two conic ſections are called *Similar Sections*, if a rectilineal figure can be inſcribed in one of them ſimilar

BOOK IV.

lar to any rectilineal figure infcribed in the other: and two conic fections are alfo called *Similar*, if a fegment can be taken of one of them fimilar to a fegment of the other.

III.

If a ftraight line touch two conic fections in the fame point, the two fections are faid to *touch one another* in the fame point.

IV.

If a circle fo touch a conic fection in any point, that no other circle, touching it in the fame point, can pafs between it and the fection, on either fide of the point of contact, it is faid to have the *fame Curvature with the Section in the point of contact*; or it is faid to be the *Ofculating Circle for that point.*

PROP. I.

Any two parabolas are fimilar to one another; and the fimilar rectilineal figures infcribed in them are to one another as the fquares of the parameters of the axes.

Fig. 117. 118. Let A B C, *a b c* be two parabolas, of which B E, *b e* are the axes, and P, *p* their parameters; and let A B C D be any rectilineal figure infcribed in the one parabola; a rectilineal figure fimilar to A B C D may be infcribed in the other, and the fimilar rectilineal figures infcribed in them are to one another as P^2 to p^2.

For draw from *b*, the vertex, the ftraight line *b a* to the curve, fo that the angle *e b a* may be equal to the angle E B A. Draw *b c* to the curve, fo that the angle *e b c* may be equal to the angle E B C. Draw B D, and draw *b d* to the curve, fo that the angle *e b d* may be equal to the angle E B D; and draw *a d*. Let A E be an ordinate to the axis B E, and let *a e* be an ordinate to the axis *b e*. Then, as the angles at E and *e* are right

right angles, the triangles ABE, *abe* are equiangular, and (4. vi.) BE : EA :: *be* : *ea*. But, by the fixth Definition Book III. BE : EA :: EA : P; and *be* : *ea* :: *ea* : *p*. We have therefore the two following ranks of magnitudes, in which the magnitudes taken two and two in the fame order have the fame ratio to one another;

BE : EA : P
be : *ea* : *p*

and confequently (22. v.) BE : P :: *be* : *p*; or by alternation BE : *be* :: P : *p*. But (4. vi. and altern.) BE : *be* :: BA : *ba*, and therefore (11. v.) BA : *ba* :: P : *p*. In the fame way it may be proved, that BC : *bc* :: P : *p*, and that BD : *bd* :: P : *p*; and therefore (11. v.) BC : *bc* :: BD : *bd*, and by alternation BC : BD :: *bc* : *bd*. But as the angles EBD, *ebd* are equal, and the angle EBC equal to the angle *ebc*, the angles DBC, *dbc* are equal. Confequently (6. vi.) the triangles CBD, *cbd* are equiangular, and (4. vi. and altern.) CD : *cd* :: BC : *bc*, or P to *p*. In the fame way it may be proved that AD : *ad* :: P : *p*; and therefore the rectilineal figure *abcd* is fimilar to the rectilineal figure ABCD. The parabolas ABC, *abc* are therefore fimilar according to the firft and fecond Definitions, and it is evident (Cor. 2. 20. vi.) that the rectilineal figures ABCD, *abcd* are to one another as the fquares of their homologous fides, or (11. v.) as P^2 to p^2.

Cor. 1. It is evident from the above that fimilar rectilineal figures may be infcribed in the fimilar parabolic fegments ABCD, *abcd*, of which the homologous fides will be to one another as P to *p*, and which will be deficient from the parabolic fegments by fpaces lefs than any given. The fimilar parabolic fegments themfelves will therefore be to one another as P^2 to p^2.

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Cor. 2. In two parabolas the parameters of diameters, which contain equal angles with their ordinates, are to one another as the parameters of the axes. For, the rest remaining as above, let A G, touching the parabola A B C in A, meet the axis B E in G; and let *a g*, touching the parabola *a b c* in *a*, meet the axis *b e* in *g*. Let F be the focus of A B C, and *f* the focus of *a b c*; and draw A F, *a f*. Then, as B F is one fourth of P, and *b f* one fourth of *p*, by the above, (and 15. v.) A B : B F : : *a b* : *b f*; and therefore (6. vi.) the triangles A B F, *a b f* are equiangular, and A F : B F : : *a f* : *b f*; or, by Cor. 2. Prop. XI. Book III. (and 15. v.) the parameter of the diameter passing through A is to P as the parameter of the diameter passing through *a* to *p*. Again, by Cor. Prop. IX. Book III. the triangles A F G, *a f g* are isosceles, and, as above, the angles A F G, *a f g* are equal. The angles A G F, *a g f* are therefore equal; and, as the diameters of a parabola are parallel, the angle A G F is equal to the angle which the diameter passing through A contains with its ordinates, and the angle *a g f* equal to the angle which the diameter passing through *a* contains with its ordinates, by Prop. II. Book III.

PROP. II.

Two ellipses, or two hyperbolas, are similar to one another, if two conjugate diameters in the one be proportional to two conjugate diameters in the other, and the first two and the other two contain equal angles. On the contrary, if two ellipses, or two hyperbolas, be similar to one another, two conjugate diameters in the one will be proportional to two conjugate diameters in the other, provided the first two and the other two contain equal angles.

Let

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Let B H, $b\,b$ be two ellipfes, or two hyperbolas, and in the one let A B a diameter be to D E its conjugate as, in the other, $a\,b$ a diameter is to $d\,e$ its conjugate, and, C, c being the centers, let the angle D C B be equal to the angle $d\,c\,b$; then the ellipfe B H is fimilar to the ellipfe $b\,b$, and the hyperbola B H is fimilar to the hyperbola $b\,b$. On the contrary, if the ellipfe or hyperbola B H be fimilar to the ellipfe or hyperbola $b\,b$, and if the angle D C B contained by A B, D E, two conjugate diameters in the one, be equal to the angle $d\,c\,b$ contained by $a\,b$, $d\,e$, two conjugate diameters in the other; then A B is to D E as $a\,b$ to $d\,e$. Fig. 119. 120. 121. 122.

Part I. Let B H M L K be any rectilineal figure infcribed in the ellipfe or hyperbola B H, and from C the center draw the ftraight lines C H, C M, C L, C K. Again, at c the center of the ellipfe or hyperbola $b\,b$, make the angles $b\,c\,b$, $b\,c\,m$, $m\,c\,l$, $l\,c\,k$, equal to the angles B C H, H C M, M C L, L C K, each to each; and let the points b, m, l, k be in the curve of the fection. Then the ftraight lines $b\,b$, $b\,m$, $m\,l$, $l\,k$, $k\,b$ being drawn, the rectilineal figure $b\,b\,m\,l\,k$ infcribed in the ellipfe $b\,b$ is fimilar to B H M L K infcribed in the ellipfe B H; and the rectilineal figure $b\,b\,m\,l\,k$ infcribed in the hyperbola $b\,b$ is fimilar to B H M L K infcribed in the hyperbola B H. For draw H G an ordinate to A B, and $b\,g$ an ordinate to $a\,b$. Then, as the angles D C G, $d\,c\,g$ are equal, and as, by Cor. 2. Prop. IV. Book II. H G is parallel to D C, and $b\,g$ parallel to $d\,c$, the angles (29. I.) at G and g are equal. The angles H C G, $b\,c\,g$ are alfo equal, by conftruction, and therefore the triangles H C G, $b\,c\,g$ are equiangular. Confequently (4. vi.) $GH : CG :: gb : cg$; and $GH^2 : CG^2 :: gb^2 : cg^2$, and (16. v.) $GH^2 : gb^2 :: CG^2 : cg^2$. But, by hypothefis, (and 15. v.) $CB : CD :: cb : cd$, and therefore $CB^2 : CD^2 :: cb^2 : cd^2$; and,

 by

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by Prop. V. Book II. $CB^2 : CD^2 :: AG \times GB : GH^2$; and $cb^2 : cd^2 :: ag \times gb : gb^2$. Hence (11. v.) $AG \times GB : GH^2 :: ag \times gb : gb^2$; and (16. v.) $AG \times GB : ag \times gb :: GH^2 : gb^2$. Consequently (11. v.) $AG \times GB : ag \times gb :: CG^2 : cg^2$; and therefore (16. v.) $AG \times GB : CG^2 :: ag \times gb : cg^2$, and (by 5. and 6. ii. and 17. and 18. v.) $CB^2 : CG^2 :: cb^2 : cg^2$. We have therefore (22. vi.) $CB : CG :: cb : cg$; and (16. v.) $CB : cb :: CG : cg$. Again, by the similar triangles, $CG : cg :: CH : ch$; and therefore (11. v.) $CB : cb :: CH : ch$, and by alternation $CB : CH :: cb : cb$. Consequently (6. vi.) the triangles BCH, bcb are similar. In the same way it may be proved that $CB : cb :: CM : cm$; and therefore (11. v.) that $CH : cb :: CM : cm$. Consequently, by alternation, (and 6. vi.) the triangles MCH, mcb are similar; and in the same way it may be proved, that the triangle lcm is similar to the triangle LCM, the triangle lck to the triangle LCK, and the triangle kcb to the triangle KCB. The rectilineal figures BHMLK $bbmlk$ (20. vi.) are therefore similar, and the sections BH, bb are therefore similar, according to the first and second Definitions.

Part II. If ab be not to de as AB to DE, let AB be to DE as ab to a straight line greater or less than de; and with this straight line as a conjugate diameter, and AB as a transverse diameter, suppose an ellipse or hyperbola to be described. Then this ellipse or hyperbola will fall without or within the section bb of the same name, and, by the preceding part, a rectilineal figure may be inscribed in the section, at present supposed to be described, similar to the rectilineal figure BHMLK. But as, by hypothesis, the sections BH, bb are similar, a rectilineal figure may be inscribed in the section bb similar to the rectilineal figure BHMLK,

ac-

according to the firſt and ſecond Definitions. Let this inſcribed figure be *b b m l k*. Then (21. vi.) in the ſection, having *a b* for its tranſverſe diameter, and falling either without or within the ſection *b b*, a rectilineal figure may be inſcribed ſimilar to *b b m l k*; which (from Def. i. vi. and 21. i.) is evidently abſurd. Conſequently, the ſections B H, *b b* being ſimilar, and the angles D C B, *d c b* equal, A B is to D E as *a b* to *d e*.

Cor. 1. In ſimilar ellipſes, or ſimilar hyperbolas, diameters which contain equal angles with the axes are to one another as the axes. For if A B, *a b* be the tranſverſe, and D E, *d e* the conjugate axes, then the angles B C H, *b c h* being equal, C B is to C H as *c b* to *c h*, as was above demonſtrated.

Cor. 2. From the above (and Cor. 2. 20. vi.) it is evident that ſimilar rectilineal figures may be inſcribed in ſimilar ellipſes, or in ſimilar hyperbolic ſegments, which ſhall be to one another as the ſquares of the tranſverſe, or as the ſquares of the conjugate axes.

Cor. 3. Similar ellipſes, and ſimilar hyperbolic ſegments, are to one another as the ſquares of their tranſverſe, or as the ſquares of their conjugate axes. This is evident from the preceding Cor. (and 2. xii.) as a rectilineal figure may be inſcribed in an ellipſe, or in a hyperbolic ſegment, which ſhall be deficient from the ellipſe, or hyperbolic ſegment, by a ſpace leſs than any given ſpace.

Cor. 4. The angles contained by the aſymptotes of ſimilar hyperbolas are equal to one another; and if the angles contained by the aſymptotes of two hyperbolas be equal, the hyperbolas will be ſimilar. For let A B, D E be the axes, C the center, and C S an aſymptote of the hyperbola B H; and let *a b*, *d e* be the axes, *c* the center, and *c s* an aſymptote of the hyperbola *b h*. Let B S touch the hyperbola B H in the vertex B, and meet

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the asymptote in S; and let bs touch the hyperbola bb in the vertex b, and meet the asymptote in s. Then the angle C B S is equal to the angle cbs, as each is a right one, and, by Cor. 3. Prop. XV. Book III. B S is equal to C E, and bs is equal to ce; and therefore, if the hyperbolas be similar, by the second part of this Prop. (and 15. v.) C B : B S : : cb : bs. Consequently (6. vi.) the angle B C S is equal to the angle bcs. On the contrary, if the angle B C S be equal to the angle bcs, then (4. vi.) C B : B S : : cb : bs; and therefore, by the first part of this Prop. the hyperbolas B H, bh are similar.

PROP. III.

If two ellipses, or two hyperbolas, be similar, the transverse axis in the one will be to the distance between the foci as the transverse axis in the other to the distance between the foci. On the contrary, two ellipses, or two hyperbolas, will be similar, if the transverse axis in the one be to the distance between the foci as the transverse axis in the other to the distance between the foci.

Fig. 119. 120. 121. 122.

Part I. Let B H, bh be two similar ellipses, and B H, bh be two similar hyperbolas. Let A B, D E be the transverse and conjugate axes, C the center, and F, O the foci of the one; and let ab, de be the transverse and conjugate axes, c the center, and f, o the foci of the other; then A B is to F O as ab is to fo.

For in the ellipses draw E O, eo, but in the hyperbolas draw E A, ea. Then, by Cor. 2. and Cor. 3. Def. XI. Book II. in the ellipses E O is equal to C A. and eo is equal to ca; but in the hyperbolas E A is equal to C O, and ea is equal to co. Hence, as the section B H is similar to the section bh, by the second part of Prop. II. in the ellipses E O : E C : : eo : ec; but

but in the hyperbolas C A : C E : : $c\,a : c\,e$. Confequently, as the angles at C and c are equal, being right angles, the triangles (7. vi.) E C O, $e\,c\,o$ in the ellipfes are equiangular; and in the hyperbolas the triangles (6. vi.) A C E, $a\,c\,e$ are equiangular. Hence in the ellipfes E O, or its equal C A, is to C O as $e\,o$, or its equal $c\,a$, is to $c\,o$; but in the hyperbolas C A is to A E, or its equal C O, as $c\,a$ is to $a\,e$, or its equal $c\,o$.

Part II. On the contrary, the reft remaining as above, if it be allowed, either in the ellipfes or hyperbolas, that C A is to C O as $c\,a$ to $c\,o$, then it may be proved, as above, (by 7. vi.) that in the ellipfes the triangles E C O, $e\,c\,o$ are equiangular, but in the hyperbolas that the triangles A C E, $a\,c\,e$ are equiangular. Confequently in the ellipfes E O, or its equal C A, is to C E as $e\,o$, or its equal $c\,a$, to $c\,e$; and in the hyperbolas C A is to C E as $c\,a$ to $c\,e$. Hence this part of the Cor. is evident from the firft part of Prop. II.

Cor. If B H, $b\,h$ be two ellipfes, or two hyperbolas, of which the foci are O, F in the one, and o, f in the other, and if the triangles M O F, $m\,o\,f$ be equiangular; then if the fections be fimilar, and the point M be in the curve of B H, the point m will be in the curve of $b\,h$. For (4. vi.) O M : M F : : $om : mf$; and therefore (18. v.) in the ellipfes O M + M F : M F : : $om + mf : mf$; and (17. v.) in the hyperbolas O M − M F : M F : : $om - mf : mf$. Again, (4. vi.) either in the ellipfes or hyperbolas, M F : F O : : $mf : fo$. Confequently, in the ellipfes,

$$OM + MF : MF : FO$$
$$om + mf : mf : fo;$$

and therefore (22. v.) O M + M F : F O : : $om + mf : fo$. Alfo, in the hyperbolas,

$$OM - MF : MF : FO$$
$$om - mf : mf : fo;$$

 and

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and therefore (22. v.) OM − MF : FO :: $om - mf$: fo. But, the reſt remaining as in the Propoſition, by Prop. XIII. Book II. OM + MF in the ellipſe, but OM − MF in the hyperbola, is equal to AB, and as the ſections are ſimilar, by this Prop. AB : FO :: ab : fo. Conſequently (11. v.) in the ellipſe $ab : fo :: om + mf : fo$; and in the hyperbola $ab : fo :: om - mf : fo$. In the ellipſe, therefore, (14. v.) $om + mf$ is equal to ab, and in the hyperbola $om - mf$ is equal to ab. Conſequently the point m is in the curve of the ſection bb, by Cor. 1. Prop. XIV. Book II.

SCHOLIUM.

Having explained methods for deſcribing conic ſections, and demonſtrated the principal properties of ſimilar ſections, it may be proper, in this place, to give an explanation of ſome paſſages in the fourth ſection of the firſt Book of the Principia. The reader, who thinks ſuch explanations neceſſary, is ſuppoſed to have that juſtly celebrated work before him whilſt he peruſes this Scholium, as he is here referred to the figures which it contains. In the following explanations the Lemmas and Propoſitions of the above-mentioned Section of the Principia are printed in capital letters, to diſtinguiſh them from ſuch parts of this treatiſe as are referred to.

LEMMA XV. This is fully explained in Cor. 2. Prop. XV. Book II.

PROPOSITION XVIII. This is evident from Prop. XIII. and Cor. 2. Prop. XV. Book II.

PROPOSITION XIX. The reaſons for the deſcriptions here delivered are eaſily deduced from Cor. 3. Prop. X. and Cor. 3. Prop. VIII. Book III. and Lemma IX. and it is evident that FI will be the directrix of the parabola.

PROPO-

PROPOSITION XX. An ellipfe or hyperbola is faid to be given in fpecies when it is fimilar to a given ellipfe or hyperbola, or when the ratio of the axes to one another is given. When the ratio of the axes to one another is given, the ratio of the principal or tranfverfe axis to the diftance of the foci is alfo given, or is conftant, according to Prop. III.

The remaining part of Cafe 1. PROP. XX. is evident from Cor. 1. Prop. VIII. Book III. and Lemma IX.

Cafe 2. That a ftraight line bifecting V v at right angles will pafs through the other focus, of the ellipfe or hyperbola, has been proved in Cor. 3. Prop. XV. Book II. and that the circumference of a circle defcribed upon K k as a diameter will pafs through the fame focus has been proved in the 4th Cor. to the fame Propofition. The whole of Cafe 2. is therefore evident from the Corollaries already mentioned, and from Prop. XIII. and XV. Book II.

Cafe 3. That a circle defcribed upon K k as a diameter will pafs through the other focus is evident from Cor. 4. Prop. XV. Book II. and that V R will pafs through the fame focus is evident from Prop. XV. Book II.

Cafe 4. That $v\,b$ is equal to $a\,b$ is evident from Sir Ifaac Newton's demonftration. Recurring to proportions previoufly ftated S H : $s\,b$: : S P : $s\,p$, and therefore S H : S P : : $s\,b$: $s\,p$; and as the angles P S H, $p\,s\,b$ are equal, the triangles P S H, $p\,s\,b$ are fimilar. Again $s\,v$: $s\,p$: : $s\,b$: $s\,q$; and on account of the fimilar triangles V S P, $b\,s\,q$, S V : S P : : $b\,s$: $s\,q$. Confequently (11. V.) S V : S P : : $s\,v$: $s\,p$, and by inverfion S P : S V : : $s\,p$: $s\,v$. From the above therefore

S H : S P : S V

$s\,b$: $s\,p$: $s\,v$,

and

BOOK IV.

and (22. v.) SH : SV :: *sh* : *sv*, and as the angles VSH, *vsh* are equal, the triangles (6. vi.) VSH, *vsh* are ſimilar. From hence, and Prop. III. and its Cor. the proceedings in this caſe are evidently juſt.

LEMMA XVI. Caſe 1. By Cor. 1. Prop. VIII. Book III. PR is the directrix of the hyperbola, of which NM is the tranſverſe axis, and A, B the foci, and therefore AP is given; and by the ſame Cor. ZR : AZ :: MN : AB. Let D denote the difference between AZ, CZ, and then D by hypotheſis is given. For the ſame reaſons AQ is given, SQ being the directrix to the other hyperbola; and, as above, AZ : ZS :: AC : D. Conſequently, by the fifth Lemma, (and 1. vi.) ZR : ZS :: MN × AC : AB × D; and as MN, AC, AB, and D are given, the ratio of ZR to ZS is given. Again, Radius : ſine∠ ZTS :: TZ : ZS; and therefore by the above, and the fifth Lemma, (and 1. vi.) Radius × D : ſine∠ ZTS × AC :: TZ : AZ. But TZ and TS being given in poſition, the ſine of the angle ZTS is given, and therefore the ratio of TZ to AZ is given. The ſtraight line TA is given, and alſo the angle ATZ, and AZ : TZ :: ſine∠ ATZ : ſine∠ TAZ; and therefore the angle TAZ is given. Conſequently the triangle ATZ is given. By introducing ſome additional ſtraight lines into the figure, the triangle might be aſcertained geometrically.

The other two caſes need no explanation; nor do the remaining parts of the Section, as the ſeveral caſes in PROP. XXI. may be ſolved by the preceding LEMMA. The SCHOLIUM is evident from Prop. VIII. Book III.

PROP. IV.

If two ſtraight lines touching a conic ſection, or oppoſite hyperbolas, meet a ſtraight line which cuts the ſection,

oppo-

opposite hyperbola, or opposite hyperbolas, and is parallel to the straight line joining the points of contact, the segments of the secant between the curve or curves and the tangents will be equal to one another: and if two straight lines touching a conic section meet a straight line which touches the section, or opposite hyperbola, and is parallel to the straight line joining the points of contact, the segments of the last mentioned tangent between the point of contact and the tangents which it meets will be equal to one another. BOOK IV.

If the two tangents be parallel, this Prop. is the same as the Cor. to Prop. I. Book II. and this case has been already demonstrated. Let the two straight lines therefore A E, D E touch the conic section A D, or the opposite hyperbolas A, D, in the points A, D, and meet one another in E, and, first, let them meet in the points H, K the straight line H K, which is parallel to A D, and cuts the curve of the section, the opposite hyperbola, or the curves of the opposite hyperbolas, in the points F, G; the segments H F, K G are equal to one another. Fig. 131. 132. 133. 134.

For let the straight line E M bisect A D in M, and meet the straight line H K in N; and then, by Cor. 1. to Prop. VI. Book III. E M is a diameter, and therefore, as A D, F G are parallel, F G is bisected in N, by Cor. 1. to Prop. II. Book III. But (29. i. and 4. vi.) E M : E N :: A M : H N, and E M : E N :: D M : K N. Consequently (11. v.) A M : H N :: D M : K N, and therefore (14. v.) H N is equal to K N, and H F equal to K G.

The rest remaining as above, let the tangents A E, D E now meet the straight line L I in the points L and I, and let L I be parallel to A D, and touch the section, Fig. 131. 132. 134.

or

BOOK IV. or opposite hyperbola, in the point B; the segments L B, B I are equal.

For as L I, A D are parallel, by Prop. II. Book III. a diameter passing through B will bisect A D, and, by Prop. VI. Book III. it will pass through E. The diameter E M therefore passes through B the point of contact, and (4. vi.) E M : E B :: A M : L B. Also E M : E B :: D M : I B, and therefore (11. v.) A M : L B :: D M : I B. Consequently (14. v.) L B is equal to I B.

Fig. 131. 132. 133. 134. *Cor.* 1. If two parallel straight lines as A D, G F cut a conic section, or opposite hyperbolas, in A, D and G, F, and straight lines A G, D F joining their extremities meet in O, Q the straight line O Q, which cuts the section, or opposite hyperbolas, in P, R, and is parallel to A D, G F; the segments O R, Q P are equal. For let the tangents A E, D E meet O Q in T and S; and then (4. vi.) A H : A T :: H G : T O, and D K : D S :: K F : S Q. But, on account of the parallels, A H : A T :: D K : D S, and therefore H G : T O :: K F : S Q; and as, by this Prop. H G is equal to K F, and T R to S P, it is evident (14. v.) that O R is equal to Q P.

Fig. 131. 132. 134. *Cor.* 2. The rest remaining as above, let A G meet the tangent L I in V, and let D F meet it in Y, and the segments V B, Y B will be equal. For it may be demonstrated, as in the preceding Cor. that L V, I Y are equal; and consequently, as L B, I B are equal, the segment V B is equal to the segment Y B.

PROP. V.

If a trapezium be inscribed in a conic section, or opposite hyperbolas, and its sides be indefinitely produced, and if from any point in the curve two straight lines be drawn parallel to two adjacent sides of the trapezium and meet the opposite sides; the rectangles under the segments of these straight

ſtraight lines, between the point in the curve and the op- BOOK IV.
poſite ſides of the trapezium, will be to one another as the ſquares of the ſemidiameters parallel to them, or the ſquares of the tangents parallel to them, and meeting one another.

Let A B D C be a trapezium inſcribed in a conic ſec- Fig. 123. 124.
tion, or oppoſite hyperbolás, and let its ſides be indefinitely produced, and from any point E in the curve let the ſtraight lines E N, E H be drawn parallel to the adjacent ſides A B, A C, and let E N meet the oppoſite ſides A C, B D in Q and N, and E H meet the oppoſite ſides A B, C D in R and H; the rectangles Q E N, H E R are to one another as the ſquares of the ſemidiameters parallel to E N, E H, or the ſquares of the tangents parallel to them and meeting one another.

For if the oppoſite ſides A C, B D be not parallel, let the ſtraight lines B *d*, D F be drawn parallel to A C or H R; and let B *d* meet the curve again in *d*, and let it meet E N in *n*. Let D F meet the curve again in F, and the ſtraight line A B in G. Draw the ſtraight line C *d*, and let it meet the ſtraight line H R in *h*, and the ſtraight line D F in S. Let the ſtraight line H R meet the curve again in T. Then, by Cor. 1. Prop. IV. S D, F G are equal, and *h* T is equal to E R, and conſequently *h* E, T R are equal. The rectangle *h* E R is therefore equal to the rectangle T R E; and, on account of the parallelograms B E, A E, the rectangle Q E *n* is equal to the rectangle A R B. The rectangles T R E, A R B are therefore to one another as the rectangles *h* E R, Q E *n*; and conſequently, by Prop. V. Book. II. and Prop. XIII. Book I. theſe rectangles are to one another as the ſquares of the ſemidiameters parallel to H R, E N, or as the ſquares of tangents parallel to H R, E N, and meeting one another. Again, on account of the

BOOK IV.

the equiangular triangles H C b, D C S, H b : S D : : C b : C S; and therefore, on account of the equals and parallels, H b : F G : : E Q : A G. Alſo, on account of the equiangular triangles N B n, D B G, n B or (34. i.) E R : D G : : N n : G B; and therefore, by the fifth Lemma, H b × E R : D G × G F : : E Q × N n : A G × G B; and by alternation H b × E R : E Q × N n : : D G × G F : A G × G B. But, by Prop. V. Book II. and Prop. XIII. Book I. D G × G F and A G × G B are to one another as the ſquares of the ſemidiameters parallel to H R, E N, or as the ſquares of tangents parallel to H R, E N, and meeting one another. By the above therefore (and 11. v.) b E × E R : Q E × E n : : H b × E R : E Q × N n; and by alternation b E × E R : H b × E R : : Q E × E n : Q E × N n; and therefore (1. vi.) b E : H b : : E n : N n, and (17. and 18. v.) H E : H b : : E N : N n. Conſequently by alternation (and 1. vi.) H E × E R : E N × E Q : : H b × E R : N n × E Q; and therefore, by the above, (and 11. v.) H E × E R and E N × E Q are to one another as the ſquares of the ſemidiameters parallel to H R, E N, or as the ſquares of tangents parallel to H R, E N, and meeting one another.

Fig. 125. If the ſtraight lines A B, C D cut one another like the diagonals of a trapezium, as in Fig. 125. and the reſt be as expreſſed above in the particular enunciation; then it may be proved, as above, that Q E × E N and H E × E R are to one another as the ſquares of the ſemidiamers parallel to E N, E H, or the ſquares of the tangents parallel to them, and meeting one another.

It is evident that the method of demonſtration and concluſion will be the ſame, if one of the ſtraight lines H R, B d, D F, or even two of them, be tangents.

Fig. 123. 124. 125. *Cor.* 1. If the points A, B, C, D remain fixed, and the point E with the ſtraight lines E N, E H, always parallel

Q E N, H E R will be to one another in the ſame ratio.

Cor. 2. The points A, B, C, E remaining fixed, if the point D, the interſection of the ſtraight lines B D, C D, be carried round the ſection, or oppoſite hyperbola, in every ſituation of D the ſegments E H, E N will be to one another in the ſame ratio. For the rectangles H E R, Q E N in every ſituation of D will be to one another in the ſame ratio, and therefore as E R, Q E remain fixed, the Cor. is (I. vi.) evident.

Cor. 3. The reſt remaining as above, draw B C, and let it meet E H in L. In the ſtraight line E Q, and on the ſame ſide of E H with the point N, let the point T be taken, and let H E be to E N as L E to E T; and then B T being drawn, it will touch the ſection. For, if it be poſſible, let it meet the curve again in V; and V C being drawn, let it meet E H in X. Then, by Cor. 2. X E : E T : : H E : E N, and therefore (II. v.) L E : E T : : X E : E T, and L E, X E are equal: which is abſurd. The ſtraight line B T therefore touches the ſection. Fig. 126.

Cor. 4. Hence, the reſt remaining, if the ſtraight line B T touch the ſection in B, and meet the ſtraight line E N in T, H E will be to E N as L E to E T.

Cor. 5. If the ſtraight lines A R, C H touch the ſection in A, C, and from E, a point in the curve, E Q be drawn parallel to A R, and meet A C, the line joining the points of contact, in Q, and H E R be drawn parallel to A C, and meet the tangents in R and H; then the rectangle H E R and the ſquare of E Q will be to one another as the ſquares of the ſemidiameters parallel to H R, Q E, or as the ſquares of the tangents parallel to H R, Q E, meeting one another. For if H R meet the curve again in T, then, by Prop. IV. H T, E R are equal, and the rectangles H E R, T R E are Fig. 127.

equal:

BOOK IV. equal: and (34. i.) as E Q is equal to A R, the Cor. is evident from Prop. V. Book II. and Prop. XIII. Book I.

PROP. VI.

If a trapezium be inſcribed in a conic ſection, or oppoſite hyperbolas, and from any point in the curve a ſtraight line be drawn in a given angle to each of the ſides, or the ſides produced, the rectangle under the lines drawn to two oppoſite ſides will be to the rectangle under the lines drawn to the two other oppoſite ſides in a given ratio.

Fig. 135. 136. Let A B D C be a trapezium inſcribed in the conic ſection E A B D C, or in the oppoſite hyperbolas A D C, *e* B, and from any point E in the curve let the ſtraight lines E L, E M, E O, E K be drawn in given angles to the ſides A B, C D, B D, A C, each to each; the rectangle under E L, E M, drawn to the oppoſite ſides A B, C D, is to the rectangle under E O, E K, drawn to the other two oppoſite ſides B D, A C, in a given ratio.

For from any other point *e* in the curve of the ſection, or in the curve of either of the oppoſite hyperbolas, let the ſtraight lines *e l*, *e m*, *e o*, *e k* be drawn parallel to E L, E M, E O, E K, each to each, and to the ſame ſide of the trapezium each to each. Through the point E let the ſtraight lines Q N, H R be drawn parallel to the adjacent ſides A B, A C, and meeting the ſides A C, B D, A B, D C, in the points Q, N and R, H. Through the point *e* draw the ſtraight lines *q n*, *h r* parallel to the ſtraight lines Q N, H R, or to the ſides A B, A C, and meeting the ſides A C, B D, A B, D C in the points *q*, *n* and *r*, *h*. Then (29. i. and 4. vi.) E R : *e r* : : E L : *e l*; and E H : *e h* : : E M : *e m*. Conſequently, by the fifth Lemma, E R × E H : *e r* × *e h* : :

E L.

EL × EM : $el \times em$. Again (29. i. and 4. vi.) EQ : eq : : EK : ek; and EN : en : : EO : eo, and therefore, as before, EQ × EN : $eq \times en$: : EK × EO : $ek \times eo$. But, by Cor. 1. Prop. V. ER × EH : $er \times eh$: : EQ × EN : $eq \times en$; and therefore (11. v.) EL × EM : $el \times em$: : EK × EO : $ek \times eo$. Consequently, by alternation, the rectangle under EL, EM is to the rectangle under EK, EO in the constant or given ratio of the rectangle under el, em to the rectangle under ek, eo.

Cor. If two straight lines AR, CH touch a conic section in A, C, and if from any point E in the curve straight lines EL, EM, EK be drawn in given angles to the tangents, and AC joining the points of contact, the rectangle under EL, EM and the square of EK will be to one another in a constant or given ratio. For take any other point e in the curve, and through E, e draw HR, hr parallel to AC, and let them meet the tangents in H, R and h, r; and draw EQ, eq parallel to AR, and let them meet AC in Q, q. Then by similar triangles, as above, ER × EH : $er \times eh$: : EL × EM : $el \times em$; and by Cor. 5. Prop. V. ER × EH : $er \times eh$: : EQ^2 : eq^2. But on account of the parallel lines, the triangles EKQ, ekq are similar, and EQ^2 : eq^2 : : EK^2 : ek^2. Consequently (11. v.) EL × EM : $el \times em$: : EK^2 : ek^2. Fig. 127.

PROP. VII.

The curve of a conic section cannot meet the curve of another conic section, or the curves of opposite hyperbolas, in more than four points.

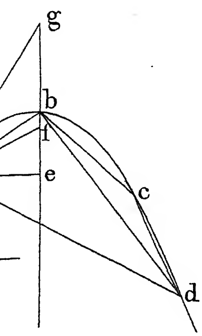
For, if it be possible, let the curve of a conic section meet the curve of another conic section, or the curves of opposite hyperbolas, in the points A, B, D, C, E; Fig. 137.

 and

BOOK IV. and draw A B, B D, D C, C A. Let the ſtraight lines E N, E H be drawn parallel to A B, A C, and let them meet B D, D C in N and H. Let the ſtraight line B F be drawn, meeting the curves again in F, I, and the ſtraight line E N in n. Let the ſtraight lines F C, I C be drawn, and let them meet the ſtraight line E H in K and L. Then, by Cor. 2. Prop. V. H E : E N :: L E : E n, and H E : E N :: K E : E n. Conſequently (11. v.) L E : E n :: K E : E n, and (14. v.) L E, K E are equal: which is abſurd. The curve of the conic ſection, therefore, does not meet the curve of the other, or the curve of oppoſite hyperbolas, in five points.

Fig. 138. *Cor.* 1. If two conic ſections touch one another, they will not meet each other in three other points. For, if it be poſſible, let the two ſections have the common tangent in the point B, and meet one another in A, E, C, and of theſe let E be the intermediate point. Let B A, B C, C A be drawn; and let E T, E H be drawn parallel to B A, A C, and let them meet B T, B C in T and H. Through the point of contact B let B D be drawn, meeting the curves in D and d, and the ſtraight line E T in N; and let D C, d C be drawn, meeting E H in I and L. Then, by Cor. 4. Prop. V. H E is to E T as L E to E N, and I E is to E N in the ſame proportion, and therefore (9. v.) I E, L E are equal: which is abſurd. The two points D, d therefore coincide, and the two ſections meet in the five points A, B, D, C, E; which by this Prop. is impoſſible.

Fig. 139. *Cor.* 2. If two conic ſections A F D, A f D touch one another in the points A, D, they will not meet one another in any other point. For, if it be poſſible, let them meet one another in the point I, and let the ſtraight line I F be drawn, meeting the tangents A B, D G in B, G, and the curve of the ſection A F D in F. As



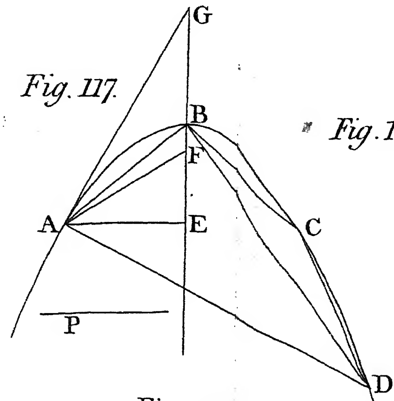
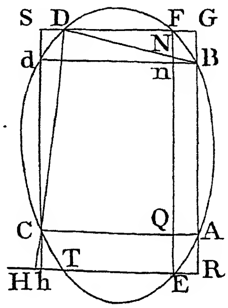


Fig. 117.



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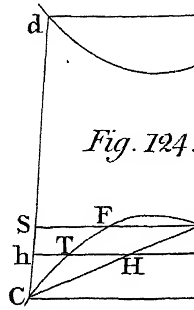

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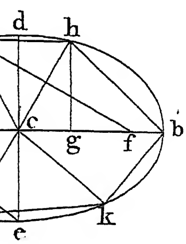
. 120.



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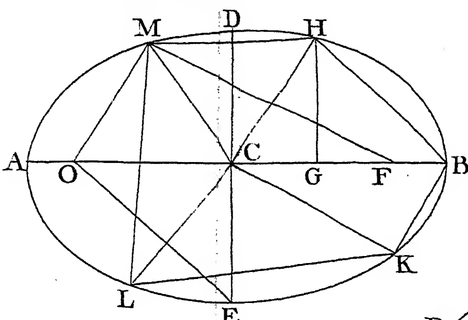

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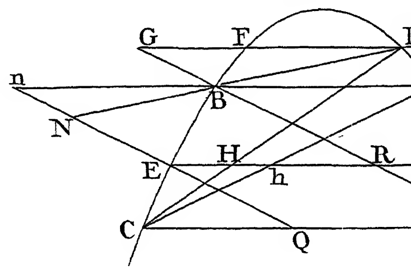

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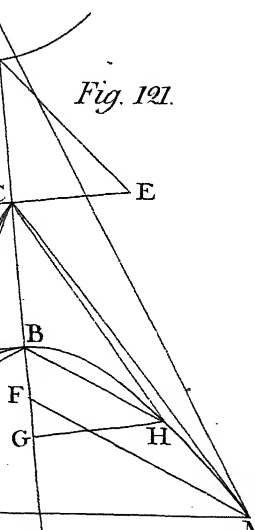

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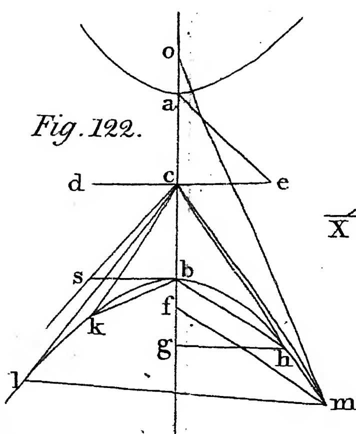

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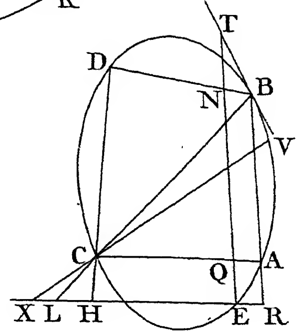
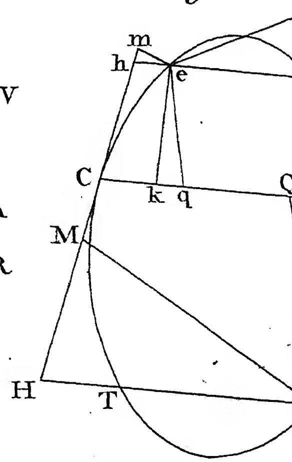
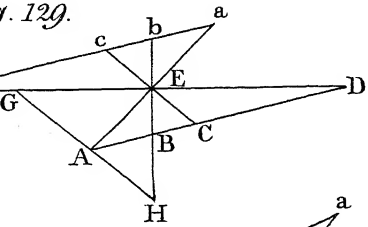


Fig. 126.

Fig. 127.





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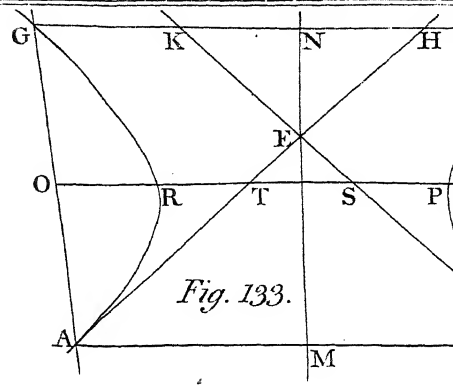

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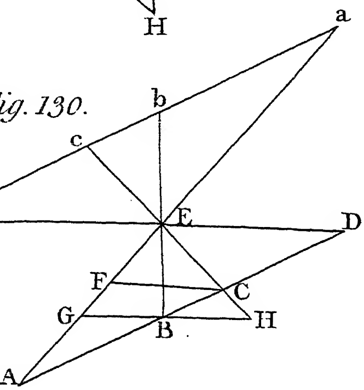

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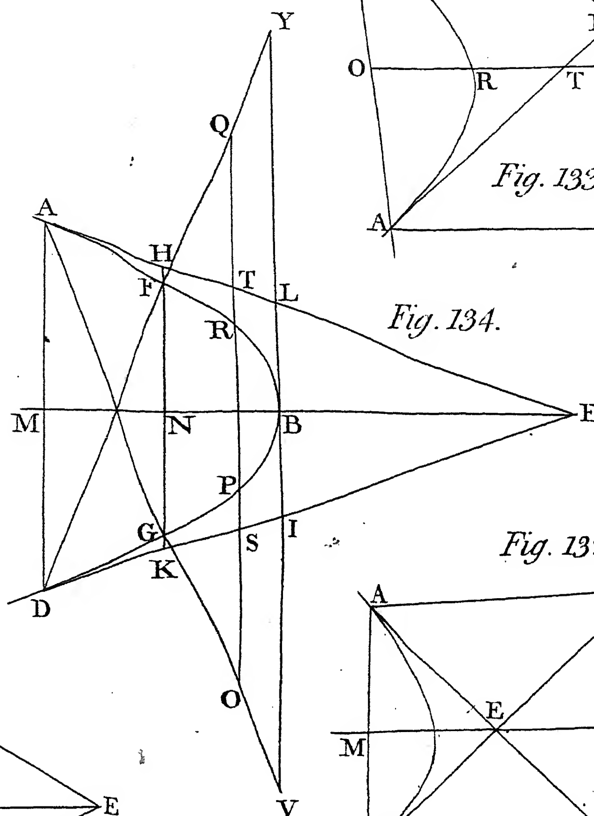

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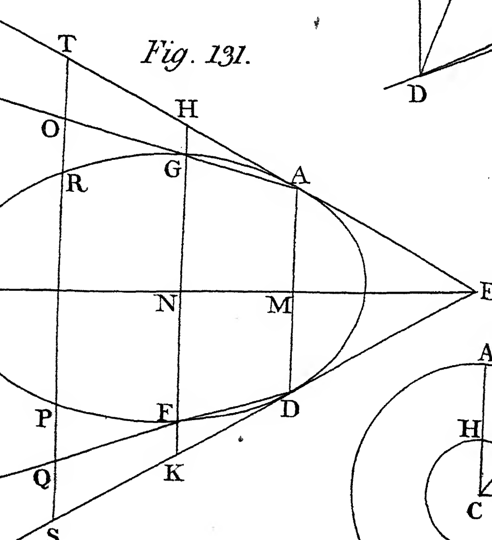

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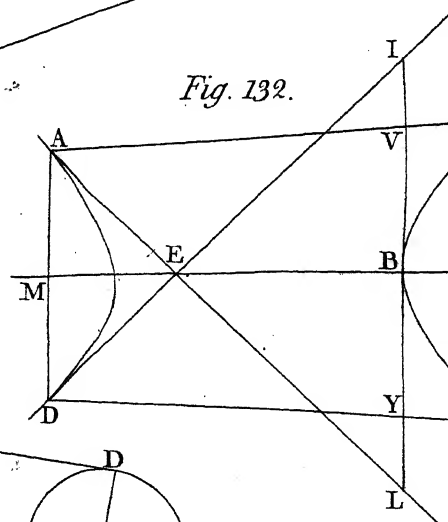

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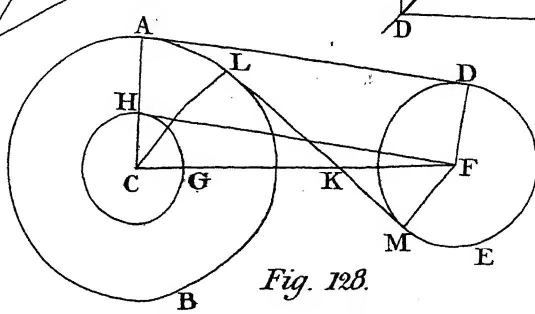

Fig. 128.

As the fections touch one another in D, by the preceding Cor. they do not meet one another in three other points. Let the ftraight line IG therefore meet the curve of the fection AfD in f. Then, by Cor. 2. Prop. XVII. Book I. $BE^2 : GE^2 :: IB \times BF : FG \times GI$; and $BE^2 : GE^2 :: IB \times Bf : fG \times GI$. Confequently (11. v.) $IB \times BF : FG \times GI :: IB \times Bf : fG \times GI$, and by alternation $IB \times BF : IB \times Bf :: FG \times GI : fG \times GI$; and therefore (1. vi.) $BF : Bf :: FG : fG$. Hence (12. v.) $BF : Bf :: BG : BG$, and therefore BF, Bf are equal: which is abfurd. Confequently the Cor. is evident.

PROP. VIII.

If a ftraight line touch a conic fection, and a ftraight line perpendicular to it be drawn through the point of contact, and meeting the axis or axes of the fection, the fegment of the perpendicular between the point of contact and the axis of a parabola, or between the point of contact and the tranfverfe axis of the fection, will be the leaft of all ftraight lines which can be drawn from the fame point in the axis, and on the fame fide of it, to the curve; but the fegment of the perpendicular between the point of contact and the conjugate axis of the ellipfe will be the greateft of all ftraight lines which can be drawn from the fame point in the axis, and on the fame fide of it, to the curve.

Let the ftraight line PR touch a conic fection in the point P, and let the ftraight line PK, perpendicular to the tangent, meet AB the axis of a parabola, or the tranfverfe axis of the fection, in the point K, and let it meet DE the conjugate axis of the ellipfe in M; the fegment PK is the leaft of all the ftraight lines which can be drawn from K, on the fame fide of AB, to the Fig. 140. 141. 142.

 curve;

BOOK IV.

curve; and in the ellipse the segment M P is the greatest of all the straight lines which can be drawn from M, on the same side of D E, to the curve.

Let the tangent P R meet the axis A B in R; and draw P G a double ordinate to A B, and let it meet A B in F, and the curve again in G, and draw R G. Then, as the angles at F are right angles, and as P F is equal to F G, we have (4. i.) P R equal to G R, G K equal to P K, and K G R a right angle, being equal (8. i.) to the angle K P R. Let the straight line B H, touching the section in the vertex B, meet the tangent P R in H, and draw K H. Then in the parabola the square of
Fig. 140. B H is to the square of P H, as the parameter of the axis A B to the parameter of the diameter passing through P, by Prop. IV. Book III. and therefore, by Cor. 1. Prop. XI. Book III. the square of B H is less than the square of P H. Consequently, as K H is common to the two right angled triangles K B H, K P H, the square of K B (47. i.) must be greater than the square of K P, and K P must be less than K B. Again,
Fig. 141. 142. in the ellipse or hyperbola, C being the center, by Prop. V. Book II. the square of B H is to the square of P H as the square of C D to the square of the semidiameter parallel to P H; and therefore, by Prop. XI. Book II. the square of B H is less than the square of P H. Consequently, for the same reasons as in the parabola, P K is less than K B. If therefore with K as a
Fig. 140. 141. 142. center, and K P as a distance, in each section, a circle be described, its circumference will pass through G, and cut A B within the section*; and as K P R, K G R are right angles, P R and G R (16. iii.) are tangents to

* The circle is intentionally omitted in the figures. The description of it would have made them more complex, and not rendered the Proposition or either of the Corollaries more perspicuous.

the

VII. the circumference of the circle cannot meet the curve of the ſection in any other point beſides P and G; and therefore K P is the leaſt of all ſtraight lines which can be drawn from K, on the ſame ſide of A B, to the curve.

In the ellipſe draw P N a double ordinate to the conjugate axis D E, and let it meet D E in L, and the curve again in N. Let the tangent P R meet D E in T, and draw N T, N M. Then it may be proved, as above, that M N is equal to M P, N T equal to P T, and that the angle M N T is a right angle, being equal to the angle M P T. In the ellipſe let the ſtraight line D V, touching the ſection in the vertex D, meet the tangent P R in V, and draw M V. Then, by Cor. 3. Prop. III. Book II. D V is parallel to the axis A B, and by Prop. V. Book II. the ſquare of D V is to the ſquare of P V, as the ſquare of C B to the ſquare of the ſemidiameter parallel to P V; and therefore, by Prop. XI. Book II. the ſquare of D V is greater than the ſquare of P V. Conſequently, as the angles at D and P are right angles, and M V common to the two triangles M D V, M P V, the ſquare of M P (47. i.) muſt be greater than the ſquare of M D. If therefore with M as a center, and M P as a diſtance, a circle be deſcribed, its circumference will paſs through N, it will cut D E without the ellipſe, and, for the ſame reaſons as above, the ſtraight lines T P, T N will touch it and the ſection in the points P, N. Conſequently, by Cor. 2. Prop. VII. the circumference of the circle cannot meet the curve of the ſection in any other point beſides P and N; and therefore M P is the greateſt of all ſtraight lines which can be drawn from M, on the ſame ſide of D E, to the curve of the ellipſe. Fig. 141.

Cor. 1. If a ſtraight line as P G be a double ordinate

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Fig. 140. 141. 142. to A B the axis of a parabola, or the tranſverſe axis of a conic ſection, a circle touching the ſection in P, and paſſing through G, will alſo touch the ſection in G; and the other parts of its circumference will fall wholly within the ſection. For P G will be in the circle, and, being biſected by A B at right angles, the center of the circle (Cor. 1. iii.) will be in A B. Let K be the center, and draw K P, and let P R be common to the circle and ſection, according to the third Definition. Then (18. iii.) K P R is a right angle; and K G, G R being drawn, it may be proved, as above, that the angles K G R, K P R are equal. Conſequently G R will touch the circle, (16. iii.) and it is evident, from Cor. 2. Prop. VI. Book III. that it alſo touches the ſection. Hence the Cor. is manifeſt.

Fig. 141. 142. *Cor.* 2. If a ſtraight line as P N be a double ordinate to D E the conjugate axis of an ellipſe, or of oppoſite hyperbolas, a circle touching the ellipſe or hyperbola B P in P and paſſing through N will alſo touch the ellipſe or the oppoſite hyperbola in N. For let the axis D E meet the common tangent P R in T, and the ordinate P N in L. Then, as P N will be in the circle, the center of the circle (Cor. 1. iii.) will be in D E. Let M be the center, and draw M P, M N, N T. Then (4. i.) M P is equal to M N, and T P equal to T N; and therefore (8. i.) the angle M N T is equal to the angle M P T, which is a right one. Hence (16. iii.) the circle touches the ellipſe or the oppoſite hyperbola in N, and N T is the common tangent to the circle and ſection.

In this caſe it is evident, that the circle deſcribed with the center M, and the diſtance M P, falls without the ellipſe, and without each of the oppoſite hyperbolas. For, by this Prop. M P in the ellipſe is the greateſt ſtraight line which can be drawn from M to the curve; and in the hyperbolas, as T P, T N are the common tangents, it is evident that M P, M N are the leaſt ſtraight

If from the vertex of the axis of a parabola, or from a vertex of the tranſverſe axis of an ellipſe or hyperbola, a ſegment be taken in the axis equal to its parameter, a circle deſcribed about this ſegment as a diameter will fall wholly within the ſection; but if from a vertex of the conjugate axis of an ellipſe a ſegment be taken in the axis equal to its parameter, a circle deſcribed about this ſegment as a diameter will fall wholly without the ſection.

Let A B be an axis of a conic ſection, and in the hyperbola the tranſverſe axis, and from the vertex A let the ſegment A C be taken in the axis equal to its parameter; the circle A E C deſcribed about A C as a diameter will fall wholly within the ſection, unleſs A B be the conjugate axis of the ellipſe, and if A B be the conjugate axis of the ellipſe, the circle will fall wholly without the ſection. Fig. 143. 144. 145. 146.

For through A draw the ſtraight line A D equal to A C, and at right angles to A B, and draw C D. Through any point F in A C draw F G an ordinate to A B, and let it meet the circumference of the circle in E, the curve of the ſection in G, and the ſtraight line C D in K. In the ellipſe and hyperbola draw from the vertex B the ſtraight line B D, and let it meet F G in H; but in the parabola draw D H parallel to the axis A B, and let it meet F G in H. Then, by Prop. II. Book III. A D, F K are parallel, and therefore (4. vi.) A D : A C : : F K : F C; and as A D is equal to A C, F K is equal to F C. Conſequently A F × F K is equal to A F × F C, and therefore (35. iii.) A F × F K is equal to the ſquare of

 E F.

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E F. But, by Cor. 1. Prop. VI. Book II. and Prop. III. Book III. the ſquare of F G is equal to A F × F H. In the parabola and hyperbola therefore, and when A B is the tranſverſe axis of the ellipſe, the ſquare of F G is greater than the ſquare of F E, and conſequently the point G is without the circle. But if A B be the conjugate axis of the ellipſe, as in Fig. 146. the ſquare of F E will be greater than the ſquare of F G, and therefore the point G will be within the circle.

PROP. X.

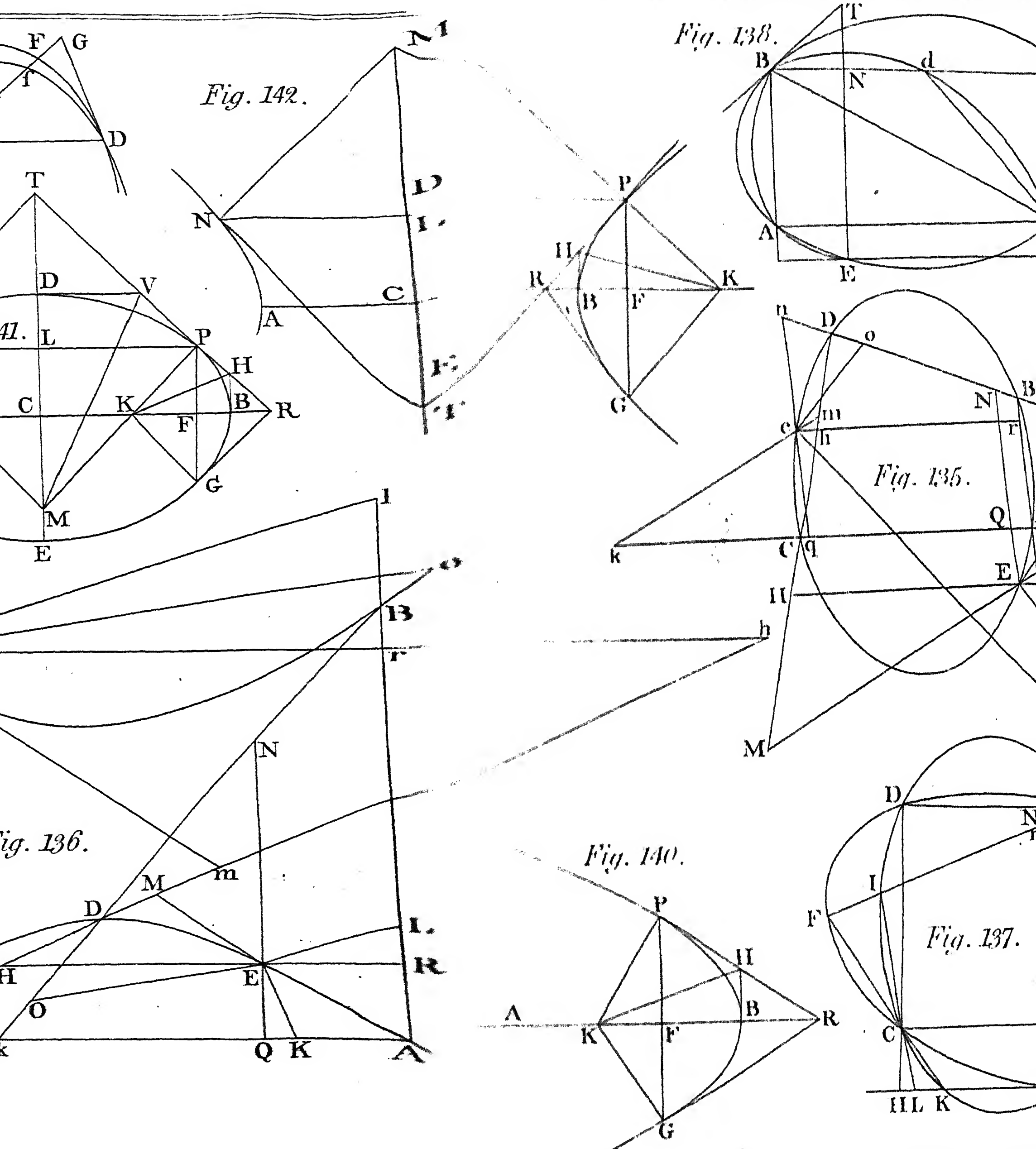
If from the vertex of the axis of a parabola, or from a vertex of the tranſverſe axis of an ellipſe or hyperbola, a ſegment be taken in the axis greater than its parameter, a circle deſcribed about this ſegment as a diameter will fall without the ſection on each ſide of the vertex; but if from a vertex of the conjugate axis of an ellipſe a ſegment be taken in the axis leſs than its parameter, a circle deſcribed about this ſegment as a diameter will fall within the ellipſe on each ſide of the vertex.

Fig. 147. 148. 149.

Firſt, let A B be the axis of a parabola, or the tranſverſe axis of an ellipſe or hyperbola, and from the vertex A let the ſegment A C be taken in the axis greater than its parameter; the circle A I C deſcribed about A C as a diameter will fall without the ſection on each ſide of the vertex A.

In the ellipſe let the point C be between the vertices A, B, and in each ſection let the ſtraight line A D be drawn perpendicular to the axis A B, and equal to its parameter. In A D let the ſegment A F be taken equal to A C, and draw C F. In the ellipſe and hyperbola let the ſtraight line B D be drawn, and let it meet C F in E; but in the parabola let the ſtraight line D E be drawn parallel to the axis, and let it meet C F in E. In each

Plate.



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B

each ſection let any point G be taken in C F, between the points E, F; and draw G L parallel to the ordinates of A B, and let it meet D E in H, the circumference of the circle in I, the curve of the ſection in K, and the axis A B in L. Then, by Prop. III. Book III. and Cor. 1. Prop. VI. Book II. the ſquare of the ordinate L K is equal to the rectangle under A L, L H; and on account of the equals A F, A C, the rectangle under A L, L G is equal to the rectangle under A L, L C, and therefore (35. iii.) equal to the ſquare of I L. But the rectangle under A L, L G is greater than the rectangle under A L, L H, and therefore the ſquare of I L is greater than the ſquare of L K. Conſequently the point I is without the ſection; and if the ſtraight line G L meet the circumference of the circle again in M, the point M, for the ſame reaſons as above, will be without the ſection. The circle A I C therefore falls without the ſection on each ſide of the vertex A.

Secondly, let A B be the conjugate axis of an ellipſe, and from the vertex A let the ſegment A C be taken in the axis leſs than its parameter; the circle A I C deſcribed about A C as a diameter will fall within the ellipſe on each ſide of the vertex A. Fig. 150.

For let A C be greater than the axis A B. Let the ſtraight line A D be drawn perpendicular to A B, and equal to its parameter. In A D let the ſegment A F be taken equal to A C, and draw B D, C F, and let them meet one another in the point E. In C F take any point G between the points F, E, and draw G L parallel to the ordinates of A B, and let it meet B D in H, the curve of the ellipſe in K, the axis in L, and the circumference of the circle in I, M. Then, as above, it may be demonſtrated, that the points I, M are within the ellipſe; and therefore that the circle A I C falls within the ellipſe on each ſide of the vertex A.

Cor.

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Cor. 1. If from the vertex of the axis of a parabola, or from a vertex of the tranfverfe axis of an hyperbola, or from a vertex of either axis in an ellipfe, a fegment be taken in the axis equal to its parameter, a circle defcribed about this fegment as a diameter will have the fame curvature with the fection in the vertex. For, by the preceding Prop. in the parabola, and when A B is a tranfverfe axis, the circle A E C falls wholly within the fection, the diameter A C being equal to the parameter of the axis A B; and it is evident, that if a fegment be taken from A in A B lefs than A C, a circle defcribed about it as a diameter will fall within the circle A E C. Again, by this Prop. if from A a fegment be taken in A B greater than the parameter of A B, or greater than A C, in Fig. 147. it will fall without the fection. In Fig. 143. therefore the circle A E C has the fame curvature with the fection in A. Alfo when A B is the conjugate axis of the ellipfe, by the preceding Prop. the circle A E C falls wholly without the ellipfe, A C being equal to the parameter of A B; and it is evident, that if a fegment be taken from A in A B greater than A C, the circle defcribed about it as a diameter will fall without the circle A E C. Again, by this Prop. if from A a fegment be taken in A B lefs than the parameter of A B, or lefs than A C, in Fig. 150. a circle defcribed about it as a diameter will fall within the fection. In this cafe therefore the circle A E C in Fig. 146. has the fame curvature with the fection in A, according to the fourth Definition.

Cor. 2. If a circle touch a conic fection in the vertex of an axis, and have the fame curvature with the fection in the vertex, it will cut off from the axis a fegment equal to its parameter. This is evident from the preceding Corollary.

PROP.

PROP. XI.

If from a point in the curve of a conic ſection a double ordinate be drawn to the axis of a parabola, or to the tranſverſe axis of the ſection, and if through the point in which it meets the curve again a diameter be drawn, and from the firſt mentioned point a double ordinate be drawn to this diameter; a circle touching the ſection in the firſt mentioned point, and paſſing through the other extremity of the laſt mentioned double ordinate, will not meet the ſection in any other point beſides theſe two, and it will have the ſame curvature with the ſection in the point of contact.

From the point B in the curve of the conic ſection B A L let the ſtraight line B A be drawn a double ordinate to D H the axis of the parabola B A L, or to D H the tranſverſe axis of the ſection, and from A, the point in which it meets the curve again, draw the diameter A E, and from B draw the double ordinate B F to A E; the circle B K F, touching the ſection in B and paſſing through F, the point in which the double ordinate B F meets the curve again, will not meet the ſection in any other point beſides B, F, and it will have the ſame curvature with the ſection in the point B. Fig. 153. 154. 155. 156.

For if the circle K F B paſs through the point A, it will touch the ſection alſo in A, by Cor. 1. Prop. VIII. and the other parts of the circumference will fall wholly within the ſection; and if the ſection be an ellipſe, and B E be drawn, it will be an ordinate to the conjugate axis; and therefore if the circle K F B paſs through E, it will alſo touch the ſection in E, by Cor. 2. Prop. VIII. and the other parts of the circumference will fall wholly without the ellipſe. If therefore the circle K F B paſs through A in any ſection, or through

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through E in the ellipse, it will touch the section in the points B, A or B, E, and, by Cor. 2. Prop. VII. it will not pass through F, contrary to hypothesis. Hence it is evident that the circle K B F in any section is greater than a circle which touches the section in B, A, but in the ellipse it must be less than a circle which touches the ellipse in B, E; and therefore, in any section, the circumference of the circle K B F meets the straight line B A without, but the straight line B E within the section. Let the circumference meet B A in K, and A E in Q. If it be possible, let the circumference of the circle meet the section in L. Draw the common tangent B D, meeting the axis D H in D, and draw A D. Then, by Cor. 2. Prop. VI. Book III. A D will touch the section, and therefore, by Prop. II. Book III. A D, B F are parallel, and on account of the axis D H, the tangents (4. i.) A D, B D are equal. Let L M be drawn parallel to A D or F B, meeting the curve again in I, and the tangent B D in M. Then, by Prop. XIII. Book I. $BD^2 : AD^2 :: BM^2 : LM \times MI$; and therefore BM^2 is equal to $LM \times MI$; and as, by hypothesis, the point L is in the circumference of the circle, the point I (36. iii.) is also in the circumference of the circle. The circle K F B therefore, touching the section in B, meets the section in F, L, I, which, by Cor. 1. Prop. VII. is impossible. The circle K F B therefore does not meet the section in any point besides B, F; and as the point K is without, and the point Q within the section, the arch B K F will be without, and the arch F Q B will be within the section.

The circle K F B will also have the same curvature with the section in the point B. For let any other circle as B R L, touching the section in B, be described; and, first, let B R L be less than K F B. Then, as the straight

iii.) will be in the ſame ſtraight line, and therefore the leſſer circle B R L will fall wholly within the circle K F B. The circle B R L therefore cannot paſs between the arch B Q F and the curve of the ſection; and if it paſs through A, or be leſs than a circle paſſing through A, it will fall wholly within the ſection, by Cor. 1. Prop. VIII. and therefore it cannot paſs between the arch F K B and the curve of the ſection. But let the circle B R L be greater than the circle paſſing through A, and meet the ſtraight line B A in V, and the curve of the ſection in L*. Let the ſtraight line L M be drawn parallel to A D, meeting the curve of the ſection again in I, and the tangent B D in M. Then, by Prop. XIII. Book I. $BD^2 : AD^2 :: BM^2 : LM \times MI$, and, on account of the equals B D, A D, the ſquare of B M is equal to $LM \times MI$. The point I therefore (36. iii.) is in the circumference of the circle, and conſequently, by Cor. 1. Prop. VII. it meets the ſection only in the points B, L, I. Again, it is evident that the circle B R L meets B F within, and B A without the ſection, and therefore that the curve of the ſection between the points F, L are without the circle. In the tangent B M therefore take any point T between B and M, and let the ſtraight line T P be drawn parallel to A D, meeting the ſection in P, s, and the circle B R L in R, p. Then, as above, it may be demonſtrated that BT^2 is equal to $PT \times Ts$. But the ſquare of B T (36. iii.) is equal to the rectangle $RT \times Tp$, and therefore the rectangles $PT \times Ts$, $RT \times Tp$ are equal. Conſequently $PT : TR :: Tp : Ts$, and as P T is greater than T R, Tp is greater than Ts. The arch BpI therefore falls within the ſection. Fig. 153. 154. 156.

* Such lines as could eaſily be ſupplied by the mind of the reader are omitted in Fig. 155. and 156.

Se-

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Fig. 155. Secondly, let the circle B R L be greater than the circle K F B; and it may then be demonſtrated, as above, that the circle B R L falls without K F B, and therefore that it cannot paſs between the arch F K B and the ſection. Again, if the circle B R L paſs through E in the ellipſe, or be greater than the circle paſſing through E, it will fall wholly without the ellipſe, and therefore it will not paſs between the ellipſe and the arch F Q B. But in any ſection let the circle B R L meet A E within the ſection, and the curve of the ſection in L. Draw the ſtraight line L M parallel to A D, and let it meet the ſection in I, and the tangent B D in M. In B D take any point T between B and M. Draw T R, and let it meet the ſection in P and *s*, and the circle B R L in R and *p*. Then, as above, it may be demonſtrated that the arch P I is without the ſection. The circle K F B therefore has the ſame curvature with the ſection in the point of contact B, according to Def. IV.

Cor. From the above, and Cor. 1. Prop. X. it is evident that only one circle, touching a conic ſection in a given point, can have the ſame curvature with the ſection in that point.

PROP. XII.

If a circle touch a conic ſection, and have the ſame curvature with the ſection in the point of contact, it will cut off from the diameter of the ſection paſſing through the point of contact a ſegment equal to its parameter.

If the point of contact be the vertex of an axis, the Propoſition has been demonſtrated, as ſtated in Cor. 2. Prop. X. but if the circle K B F touch the ſection in the

Fig. 160. 161. 162. point B, which is not a vertex of an axis, let every thing remain as in the preceding Prop. and let K B F be the circle having the ſame curvature with the ſection in the

the point B. Let the circumference of the circle KBF therefore meet BC, drawn through B the point of contact and C the center, in the point N, if the section be an ellipse or hyperbola; or let it meet the diameter BN in the point N, if the section be a parabola; in either case the segment BN is equal to the parameter of the diameter BCN. BOOK IV.

First, let the section be an ellipse or hyperbola, and draw the diameter CV parallel to the tangent DA, and the diameter TM parallel to the tangent BD, and meeting BF in G. Let the tangent DA meet the diameter TM in M, and the diameter BC in L. Let the diameter AE meet its ordinate BF in I. Then, on account of the axis DC and its ordinate AB, the semidiameters CA, CB are equal; and by similar triangles LA : AC :: BI : CI; and AC : AM :: CI : IG. Consequently, Fig. 160. 161.

LA : AC : AM

BI : CI : IG,

and (22. v.) LA : AM :: BI : IG; and therefore (22. vi.) LA × AM : BI × IG :: LA^2 : BI^2, or, by the above, as AC^2 to CI^2. But, by Cor. 2. Prop. IV. Book II. the diameters TM, BL are conjugate, and therefore LA × AM is equal to CV^2, by Cor. 2. Prop. IX. Book II. Consequently CV^2 : BI × IG :: AC^2 : CI^2; and, by alternation, CV^2 : AC^2 :: BI × IG : CI^2; and, by Prop. V. Book II. CV^2 : AC^2 :: BI^2 : AI × IE. Consequently, by the tenth Lemma, (and 12. v. and 3. ii.) CV^2 : AC^2 :: GB × BI : AC^2, and therefore (14. v.) CV^2 is equal to GB × BI. But, by Prop. V. Book II. AD^2 : BD^2 :: CV^2 : CT^2; and therefore, on account of the equals AD, BD, the square of CV is equal to the square of CT. The square of CT is therefore equal to GB × BI; and NB × BC is equal to FB × BG, by the seventh Lemma. If therefore NB be bisected in P, the rectangle under PB, BC will

BOOK IV. will be equal to G B × B I, or to C T². But C T² is equal to the rectangle under C B and half the parameter of C B. Consequently the segment B N is equal to the parameter of the diameter C B.

Fig. 162. Secondly, let the section be a parabola, and let the diameter A E meet its ordinate B F in E. Draw to the diameter B N the ordinate A C, meeting the diameter B N in C, and B F in H. Then, on account of the equals A D, B D, the parameter of the diameters A E, B C will be equal, by Prop. IV. Book III. and by Cor. Prop. V. Book III. A E, B C are equal, and as A E, B C are parallel, the triangles A E H, B C H are equiangular. Consequently (4. vi.) A E : E H : : B C : B H, and therefore (14. v.) E H, H B are equal. Consequently F B : B E : : B E : B H, and therefore F B × B H is equal to B E². But, by the seventh Lemma, F B × B H is equal to N B × B C; and as B E, A C are equal, B E² is equal to A C². The rectangle N B C therefore is equal to the square of A C. Moreover the square of A C is equal to the rectangle under B C, and the parameter of the diameter B N; and therefore the rectangle N B C is equal to the rectangle under the abscissa B C, and the parameter of the diameter B N. The segment B N is therefore equal to the parameter of the diameter drawn through B the point of contact.

Cor. 1. If from the vertex of a diameter of a parabola, or from a vertex of a transverse diameter of an hyperbola, or from a vertex of any diameter of an ellipse, a segment be taken in the diameter equal to its parameter, a circle touching the section in the vertex, and passing through the other extremity of the segment, will have the same curvature with the section in the vertex. This is evident from Cor. 1. Prop. X. and the above.

Cor. 2. If through O, the focus of the parabola, the straight

meter of the diameter B C.

For through O draw O X parallel to B D, and let it meet B C in X. Then, by Cor. Prop. XII. Book III. B X is equal to B O; and by the seventh Lemma R B × BO is equal to N B × B X. Consequently B R is equal to B N.

Cor. 3. The rest remaining as above in the ellipse and hyperbola, if the straight line B R drawn through the focus O meet the circle again in R, and X S be the transverse axis, then B R will be to the diameter C T as C T is to X S. For let the diameter C T meet B R in Y, and then, by the seventh Lemma, the rectangle R B Y is equal to the rectangle F B G. But by the above the rectangle F B G is equal to twice the square of C T, and therefore the rectangle R B Y is equal to twice the square of C T. Consequently R B : the whole diameter C T : : the semidiameter C T : B Y. But, by Cor. Prop. XVI. Book II. B Y is equal to C X, and therefore (15. v.) B R : the whole diameter C T : : the whole diameter C T : X S. Fig. 160. 161.

PROP. XIII.

If a circle touching an ellipse or hyperbola have the same curvature with the section in the point of contact, its semidiameter will be to the semidiameter of the section conjugate to that passing through the point of contact, as the square of the same semidiameter of the section to the rectangle under the semiaxes.

For if the circle touch the section in the vertex of an axis, then every thing remaining as in Prop. IX. by the Definition of a parameter, and inversion, A C is to the axis parallel to the common tangent at A, as the same axis to the axis A B. Consequently (15. v. and Fig. 144. 145. 146.

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1. iv.) as the femidiameter of the circle A E C to the femiaxis parallel to the common tangent at A, fo is the fquare of the fame femiaxis to the rectangle under the femiaxes.

Fig. 160. 161.

But if the circle of curvature do not touch the fection in the vertex of an axis, let every thing remain as in Prop. XII. and let U B be the femidiameter of the circle; and then U B will be to C T as C T² to the rectangle under the femiaxes.

For let C X be the tranfverfe femiaxis, and C H the conjugate femiaxis. From the center C draw the perpendicular C Q to the tangent B D. Let U B meet the circumference again in W, and draw W N. Then (31. iii.) the angle W N B is equal to the angle C Q B, and the angle N B W (29. i.) equal to the angle B C Q. Confequently (4. vi.) C B : C Q : : W B : B N, or (15. v.) as U B to P B; and therefore C Q × U B is equal to C B × P B. But, by Prop. XII. and the Definition of a parameter, C B × P B is equal to C T², and therefore U B : C T : : C T : C Q. Confequently (1. vi.) U B : C T : : C T² : C T × C Q. But, by Cor. 1. Prop. XIX. Book II. C T × C Q is equal to C X × C H; and therefore U B : C T : : C T² : C X × C H.

Cor. By the above the fquare of C T is equal to the rectangle under U B, C Q.

PROP. XIV.

If from the center of a circle, touching a conic fection, and having the fame curvature with the fection in the point of contact, a perpendicular be let fall upon a ftraight line drawn from the point of contact through the neareft focus, a ftraight line drawn from the point of concourfe to the point in which the diameter of the circle, paffing through the point of contact, cuts the focal axis will be at right angles to this diameter of the circle.

From

From H the center of the circle L P M, touching the conic section A P in the point P and having the same curvature with the section in P, let the perpendicular H K be drawn to the straight line P K passing through F the nearest focus; the straight line K I, drawn from K the point of concourse to I the point in which the diameter P H of the circle cuts the focal axis A I, is at right angles to H P. BOOK IV. Fig. 157. 158. 159.

For first let the section be an ellipse or hyperbola of which C is the center, A C the transverse semiaxis, and D C the conjugate semiaxis. Let N P be the common tangent, and C T the semidiameter of the section parallel to N P; and draw C G, F N perpendicular to N P. Then, by Cor. Prop. XIII. the square of C T is equal to the rectangle under H P, C G; and, by Prop. XVII. Book II. the square of C D is equal to the rectangle under I P, C G. Consequently $CT^2 : CD^2 :: HP \times CG : IP \times CG$; and therefore (1. vi.) $CT^2 : CD^2 :: HP : IP$. But, by Prop. XIX. Book II. (and 22. vi.) $CT^2 : CD^2 :: FP^2 : FN^2$; and, on account of the similar triangles, $FP^2 : FN^2 :: HP^2 : PK^2$. Consequently (11. v.) $HP^2 : PK^2 :: HP : IP$; and therefore (1. vi.) $HP^2 : PK^2 :: HP^2 : HP \times IP$, and (14. v.) PK^2 is equal to $HP \times IP$. Hence $HP : PK :: PK : IP$, and (6. vi.) P I K is a right angle. Fig. 157. 158.

Secondly, let the section be a parabola, and let the common tangent N P meet the axis A I in G. Then, by Cor. Prop. IX. Book III. G F is equal to F P, and therefore (6. i.) the angle F G P is equal to the angle F P G. But as each of the angles G P I, H K P is a right angle, the angles F P G, F P I together are equal to the angles K P I, P H K together. Consequently the angle I G P is equal to the angle P H K; and therefore (4. vi.) $GI : IP :: PK : PH$, and $HP \times PI$ is equal to $GI \times PK$. But, by Cor. 2. Prop. XII. (and 3. ii.) P K is Fig. 159.

 equal

BOOK IV. equal to half the parameter of the diameter passing through P, and therefore, by Prop. XI. Book III. PK is equal to GI. Consequently HP × PI is equal to PK^2. As above, therefore, HP : PK :: PK : IP, and (6. vi.) PIK is a right angle.

Fig. 157. 158. 159. *Cor.* If a straight line NG touch a conic section in P, and PH at right angles to it meet the focal axis AB in I, and if IK perpendicular to PH meet in the point K the straight line PL drawn through the nearest focus F, and, lastly, if KH at right angles to PL meet PH in H, the point H will be the center of the circle which touches the section, and has the same curvature with it in P.

SCHOLIUM.

Fig. 160. If a body B revolve in a space void of resistance about the center C in the curve BAF, and if the circle HKF touch the curve in the point B, and have the same curvature with it in that point; and if the straight line BCN meet the circle again in N, and CQ be perpendicular to the common tangent BQ, Sir Isaac Newton has demonstrated, in Cor. 3. to Prop. VI. Lib. I. of the Principia, that the centripetal force is reciprocally as $CQ^2 \times NB$, or directly as $\frac{1}{CQ^2 \times NB}$.

By means of this expression, and the properties of osculating circles demonstrated in the preceding Propositions, the centripal forces of bodies moving in conic sections may be easily ascertained, as in the following examples.

Fig. 160. 1. Let the body revolve in the ellipse XHS, and let the law of centripetal force tending to C the center be required.

Every thing remaining as in Prop. XIII. the centripetal force, according to the Newtonian expression, is reciprocally as $CQ^2 \times BN$. But, by Cor. 1. Prop. XIX.

XIX. Book II. $CT \times CQ = XC \times CH$, and therefore $CQ^2 = \frac{XC^2 \times CH^2}{CT^2}$; and by the Definition of a parameter, and Prop. XIII. $BN = \frac{4CT^2}{2CB} = \frac{2CT^2}{CB}$. Consequently $CQ^2 \times BN = \frac{XC^2 \times CH^2}{CT^2} \times \frac{2CT^2}{CB} = \frac{XC^2 \times 2CH^2}{CB}$; and therefore, as $XC^2 \times 2CH^2$ is a constant quantity, the centripetal force is reciprocally as $\frac{1}{CB}$, or directly as the distance CB. BOOK IV.

2. Let a body P move in an ellipse or hyperbola PA, and let the centripetal force tending to the focus F of the section be required.

The rest remaining as in Prop. XIV. let the straight line PF meet the circle again in L, and, by Cor. 3. Prop. XII. $LP : 2CT :: 2CT : 2CA$; and therefore $LP = \frac{4CT^2}{2CA} = \frac{2CT^2}{CA}$. Again, by Prop. XIX. Book II. (and 22. vi.) $FN^2 : FP^2 :: CD^2 : CT^2$, and $FN^2 = \frac{FP^2 \times CD^2}{CT^2}$. But, the Newtonian general expression being adapted to the present Figures, the centripetal force is reciprocally as $FN^2 \times LP$; and therefore this force, by the above, is reciprocally as $\frac{FP^2 \times CD^2}{CT^2} \times \frac{2CT^2}{CA} = \frac{FP^2 \times 2CD^2}{CA}$. Again, if the letter L be put for the parameter of the transverse axis, $2AC : 2CD :: 2CD : \frac{4CD^2}{2AC} = \frac{2CD^2}{AC} = L$. Consequently as L, or its value, is constant, the centripetal force is reciprocally as FP^2, or inversely as the square of the distance. Fig. 157. 158.

3. Let a body P move in the curve of a parabola PA, and let the law of centripetal force tending to the focus F be required.

 Every

BOOK IV.

Fig. 159. Every thing remaining as in Prop. XIV. let the ſtraight line P F meet the circle again in L, and let F N be drawn perpendicular to the tangent P G. Then, by Cor. 2. Prop. XII. of this, and Cor. 2. Prop. XI. Book III. $LP = 4 FP$; and, by Prop. X. Book III. $FN^2 = FP \times AF$. Again by the Newtonian general expreſſion, adapted to this Figure, the centripetal force is reciprocally as $FN^2 \times LP$. By the above therefore the centripetal force is reciprocally as $FP \times AF \times 4 FP = FP^2 \times 4 AF$. Conſeqnently, as 4 A F is conſtant, the centripetal force is reciprocally as FP^2.

PROP. XV.

If three ſtraight lines touch a conic ſection, or oppoſite hyperbolas, any one of them will be harmonically divided in its point of contact, the points in which it meets the other two, and the points in which it meets the ſtraight line joining their points of contact.

Fig. 151. 152. 165. 166. 167. Let the three ſtraight lines Q R, R P, P N touch a conic ſection Q E N, or oppoſite hyperbolas Q, N in the points Q, E, N; any one of them as R P is harmonically divided in its point of contact E, the points R, P in which it meets the other tangents, and the point A in which it meets the line Q N joining their points of contact.

Fig. 151. 152. Caſe 1. Firſt, let the tangents Q R, N P be parallel; and then, by the Cor. to Prop. XIII. Book I. $RE : EP :: QR : NP$. But (4. vi.) $QR : NP :: RA : AP$, and therefore (11. v.) $RA : AP :: RE : EP$.

Fig. 165. 166. 167. Caſe 2. Let the tangents Q R, N P meet one another in the point B; and through the point P draw P H parallel to Q R, and let it meet the curve of the ſection in G, H, and Q N in I. Then, by Prop. XIII. Book I. $RE^2 : EP^2 :: QR^2 : HP \times PG$. But, by Prop. XVII. Book I. $HP \times PG$ is equal to PI^2. Conſequently $RE^2 :$

EP^2

$EP^2 :: QR^2 : PI^2$, and (22. vi.) $RE : EP :: QR : PI$. BOOK IV.
But (4. vi.) $QR : PI :: RA : AP$, and therefore (11. v.) $RA : AP :: RE : EP$.

PROP. XVI.

If two ſtraight lines touching a conic ſection, or oppoſite hyperbolas, meet one another, a ſecant paſſing through the point of concourſe will be harmonically divided in the point of concourſe, the points in which it meets the curve or curves, and the point in which it meets the ſtraight line joining the points of contact.

Let the two ſtraight lines EA, FA, touching the ſection EDF, or the oppoſite hyperbolas E, F, in the points E, F, meet one another in A, and let the ſtraight line AB meet the curve or curves in B, D, and the ſtraight line EF in C; the ſtraight line AB is harmonically divided in the points A, D, C, B. Fig. 151. 152. 168. 169.

For through B, D draw the ſtraight lines GM, HK parallel to EF, and let them meet the tangents EA, FA in G, M, and H, K and the curve or curves in B, L, and D, I. Then, by Prop. IV. GB, LM are equal, and therefore GL is equal to BM; alſo HD, IK are equal, and therefore HI is equal to DK. By equiangular triangles GB is to HD as AB to AD, and BM, or its equal GL, is to DK, or its equal HI, in the ſame proportion. Conſequently (11. v.) $GB : HD :: LG : IH$, and by alternation $GB : LG :: HD : IH$; and therefore (22. vi.) $BG \times GL : DH \times HI :: GB^2 : HD^2$. But, on account of the parallels, $GB^2 : HD^2 :: AB^2 : AD^2$; and, by Prop. XIII. Book I. $BG \times GL : DH \times HI :: EG^2 : EH^2$. Alſo, on account of the parallels, (10. vi.) $EG^2 : EH^2 :: CB^2 : CD^2$; and therefore (11. v.) $AB^2 : AD^2 :: CB^2 : CD^2$. Conſequently (22. vi.) $AB : AD :: CB : CD$.

If AB biſect EF in C, by Cor. 1. to Prop. V. Book

 III.

BOOK IV.

III. it will be a diameter. If therefore in this case A B meet the curve of the section in two points, or the curve of each of the opposite hyperbolas in one, it must be a diameter of an ellipse, or a transverse diameter of an hyperbola; as a diameter of a parabola can meet the curve in one point only, and in the hyperbola a second diameter does not meet either of the opposite curves. Consequently, by Cor. 3. Prop. VII. Book II. A B : A D : : B C : C D.

PROP. XVII.

If two straight lines touching a conic section, or opposite hyperbolas, meet one another, and a secant pass through the point of concourse, tangents passing through the points in which the secant meets the curve or curves will either be parallel, or they will meet one another in the straight line joining the points of contact of the two first mentioned tangents.

Fig. 170. 171. 172. 173. Let the straight lines K L, K M, touching the conic section L G M, or the opposite hyperbolas L, M in the points L, M, meet one another in K, and let the straight line K B meet the curve, or opposite curves, in the points G, B; straight lines touching the curve or curves in G and B will either be parallel, or they will meet in the straight line L M.

For if the straight line K B bisect L M, by Cor. 1. Prop. VI. Book III. K B is a diameter, and L M is a double ordinate to it; and, by Prop. II. Book III. tangents passing through G, B will be parallel to L M, and therefore parallel to one another. But let K B meet L M in H, and not bisect it. Let the tangents G N, B N be drawn, and meet one another in N, and if it be possible let N not be in the straight line L M. Draw N L, and let it meet K B in V. Let the tangents N G, N B meet the tangent K L in S and T. Then, by Prop. XV.

T K

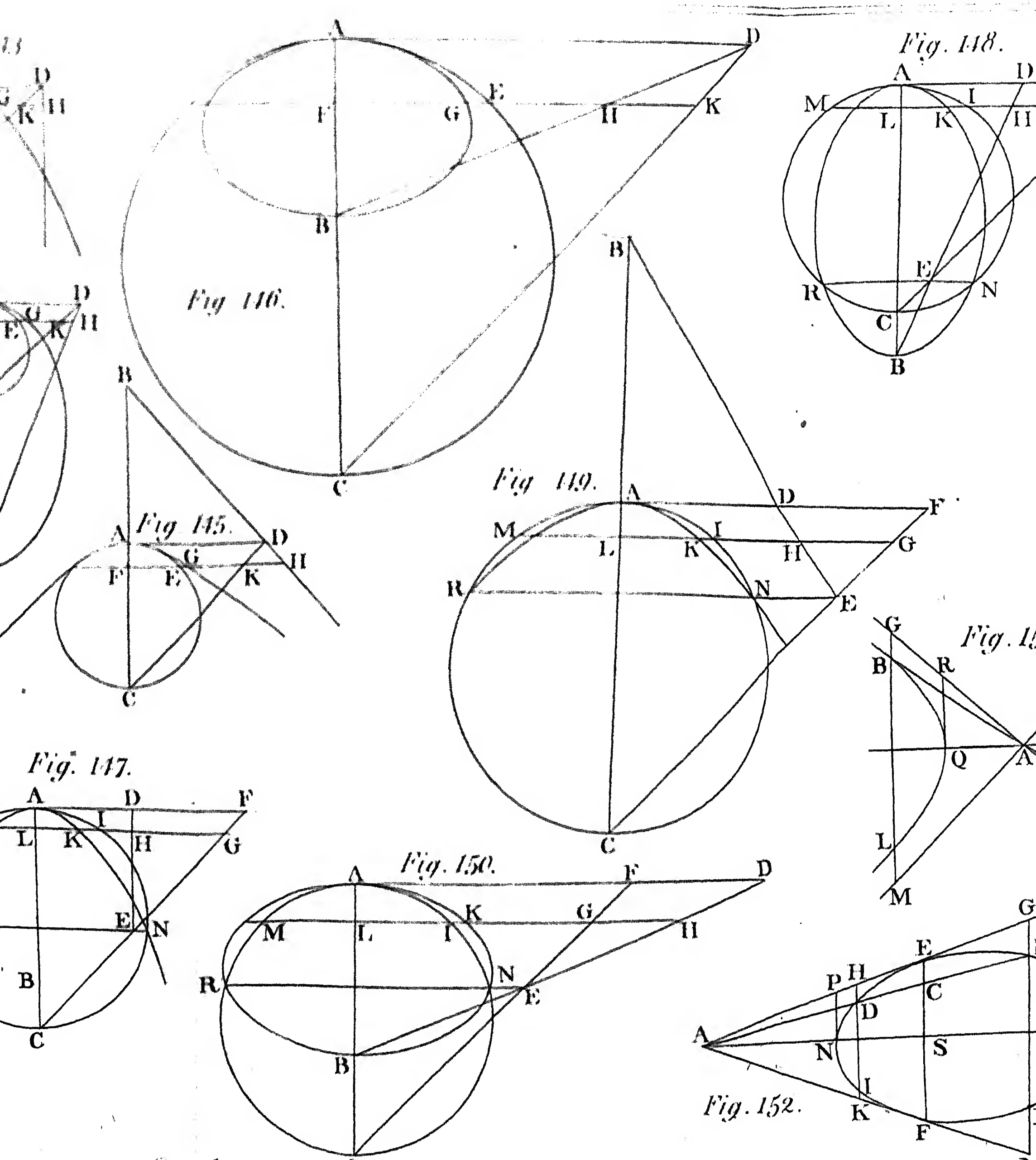


Fig. 145.

Fig. 146.

Fig. 147.

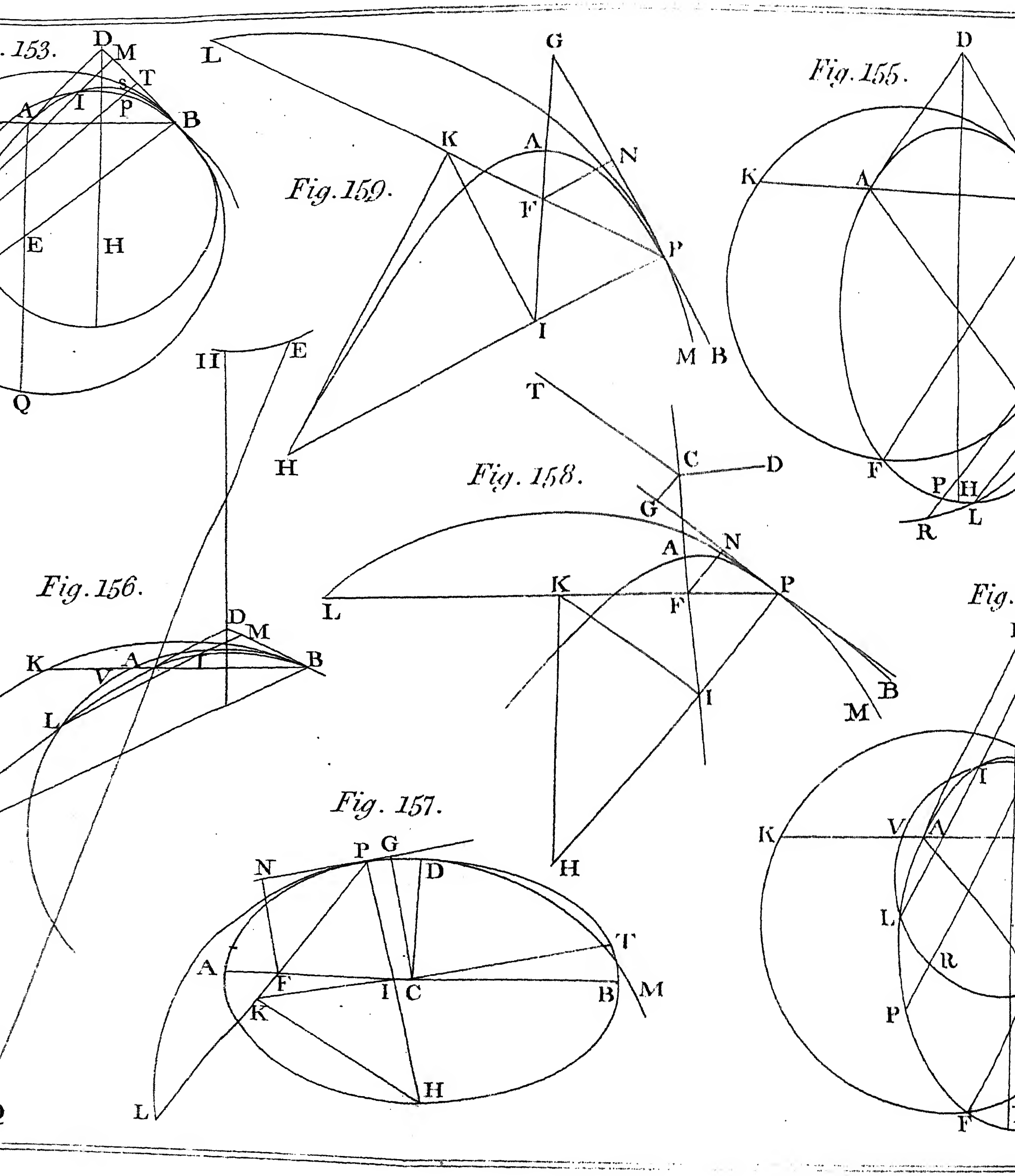
Fig. 148.

Fig. 149.

Fig. 150.

Fig. 15

Fig. 152.



.153.
D M T s I p A B E H Q
Fig. 155.
D K A F P H R L
Fig. 156.
H E D M K A V I B L
Fig. 157.
N P G D A F I C B T M K H L
Fig. 158.
T C D G A N K P L F I B M H
Fig. 159.
L G K A N F P I M B H
Fig.
K V A I L R P F

TK : KS : : TL : LS; and therefore, on account of the harmonicals NK, NG, NV, NB, by the twelfth Lemma, BK : KG : : BV : VG. But, by Prop. XVI. BK : KG : : BH : HG; and therefore (11. v.) BV : VG : : BH : HG. Confequently (17. and 18. v.) BG : VG : : BG : HG, and (14. v.) VG is equal to HG; which is abfurd. The tangents GN, BN meet therefore in the ftraight line LM.

Cor. 1. If a ftraight line as LM, not paffing through the center, cut a conic fection, or oppofite hyperbolas, in L, M, and from points N, D, &c. in LM, tangents NB, NG, DE, DF, &c. be drawn to the curve, or oppofite curves, the ftraight lines BG, FE, &c. each joining the points of contact of two tangents drawn from the fame point in LM, will meet one another in the point K, in which the tangents paffing through L, M meet one another.

Cor. 2. If the ftraight lines BG, FE, not paffing through the center, meet the curve, or oppofite curves, in B, G and F, E, and one another in K, and if they be harmonically divided in B, H, G, K and F, Q, E, K; then tangents paffing through B, G, or through F, E, will meet one another in the ftraight line drawn through H, Q.

Cor. 3. If the tangents BN, GN meet in N, and the tangents FD, ED in D, and the ftraight lines BG, FE joining the points of contact meet one another in K, and the ftraight line ND in H and Q; the ftraight lines KB, KF will be harmonically divided in K, G, H, B and in K, E, Q, F.

PROP. XVIII.

If the oppofite fides of a trapezium infcribed in a conic fection be not parallel, and neither pafs through the center, the interfection of ftraight lines joining the oppofite angular points, the interfection of two of the oppofite fides, and

BOOK IV.

and the interſections ſituated between theſe ſides, of tangents paſſing through the points in which theſe ſides meet the curve or curves, will be all four in the ſame ſtraight line.

Fig. 164. Let G B F E be a trapezium inſcribed in a conic ſection, or oppoſite hyperbolas, having no two of its oppoſite ſides parallel, and no one of them paſſing through the center; the interſection I of the ſtraight lines B E, F G joining the oppoſite angular points, the interſection of the oppoſite ſides B F, G E, and the interſections D, N, ſituated between G V, B V, of the tangents E D, F D, and G N, B N, are all four in the ſame ſtraight line.

Let the ſides B G, F E meet one another in K; and N D being drawn, let it meet K B in M, K F in L, and B F in V. Then, by Cor. 3. Prop. XVII. K B is harmonically divided in K, G, M, B, and K F in K, E, L, F. Firſt, if the ſtraight line G E do not paſs through V, draw V K, V E; and let V E meet K B in Q. Then, on account of the harmonicals V K, V Q, V M, V B, the ſtraight line K B will be harmonically divided in K, Q, M, B; which is abſurd, by Cor. 2. to the firſt Definition before Lemma XI. Conſequently B F, G E meet one another in the ſtraight line N D. Secondly, if N D do not paſs through the point I, draw I K, M I, and let M I meet K F in P. Then, on account of the harmonicals I K, G F, M P, B E, the ſtraight line K F is harmonically divided, by Lemma XII. in K, E, P, F; which, by Cor. 2. to the firſt Definition, before Lemma XI. is abſurd. Conſequently the ſtraight lines B E, G F meet one another in the ſtraight line N D; and I, V, D, N the points of the interſections are in the ſame ſtraight line N D.

SCHOLIUM.

In the following Problems, when the expreſſion Art. with

with a figure occurs, the article so numbered in the Scholium at the end of the third Book is referred to. BOOK IV.

The first six of the following solutions, although in several respects different, apply to the 59th, 60th, and 61st Problems in the Arithmetica Universalis, and also to the 22nd, 23rd, 24th, 25th, 26th, and 27th propositions in the first Book of the Principia.

PROP. XIX. PROB. I.

Given five points in the curve of a conic section, to describe the section.

Let E, A, B, C, D be five points given in the curve of a conic section, to describe the section. Fig. 163.

Draw A B, B D, D C, C A, B C; and through the point E draw E L parallel to A C, and E T parallel to A B. Let the straight line E L meet B C in L, and the straight line D C in H; and let E T meet B D in N. In E T, and on the same side of E H with N, take the segment E T so that H E may be to E N as L E to E T. Then B T being drawn it will touch the section, by Cor. 3. Prop. V. In the same way straight lines D F, A G may be drawn touching the section in D, A. Then if any two of the tangents, suppose D F, B T be parallel, the section will be an ellipse, according to Prop. VIII. Book I. and D B will be a diameter according to Cor. 1. Prop. II. Book II. In this case if a straight line be drawn from E parallel to D F, or B T, it will be an ordinate to B D; and the conjugate diameter to B D being found by art. 10. the section may be described.

But if no two of the tangents be parallel to one another, let B T meet D F in F, and A G in G. Draw F K bisecting B D in K, and G I bisecting A B in I. Then, by Cor. 1. Prop. VI. Book III. F K, G I will be dia-

BOOK IV. diameters, and therefore if they are parallel, the section will be a parabola: but if they are not parallel, they will meet in the center of the section.

If the section be a parabola let F K be bisected in P, and then it is evident, by Prop. V. Book III. and art. 9. that the section may be described. But if the section be not a parabola, let F K, G I meet in O, and O will be the center. Between F O, O K let a mean proportional O P be found, and, by Prop. VII. Book II. O P will be a semidiameter of the section, to which B D is a double ordinate. Consequently the diameter conjugate to O P being found by art 10. the section may be described.

PROP. XX.

Fig. 176. 177. *If the straight lines* A C, B D *cutting a conic section in* A, C *and* B, D, *and meeting one another in* G, *meet in* T *and* F *a straight line* T F *which touches the section in* E, *the square of* E T *will be to the square of* E F *in the ratio compounded of the ratio of the rectangle under* B G, G D *to the rectangle under* A G, G C *and of the ratio of the rectangle under* A T, T C, *to the rectangle under* B F, F D.

Fig. 176. First let the section be a parabola or hyperbola, and let the point T be on that side of the figure on which the section can be extended. Let the straight line I K be drawn parallel to B D, and let it meet the curve in I, K; and let the square of the straight line V X be equal to the rectangle I T K. Then, by Prop. XIII. Book I. the square of E T is to the square of E F as the square of V X to the rectangle B F D. But the square of V X is to the rectangle B F D in the ratio compounded of the square of V X to the rectangle A T C and of the ratio of A T C to the rectangle B F D;

that

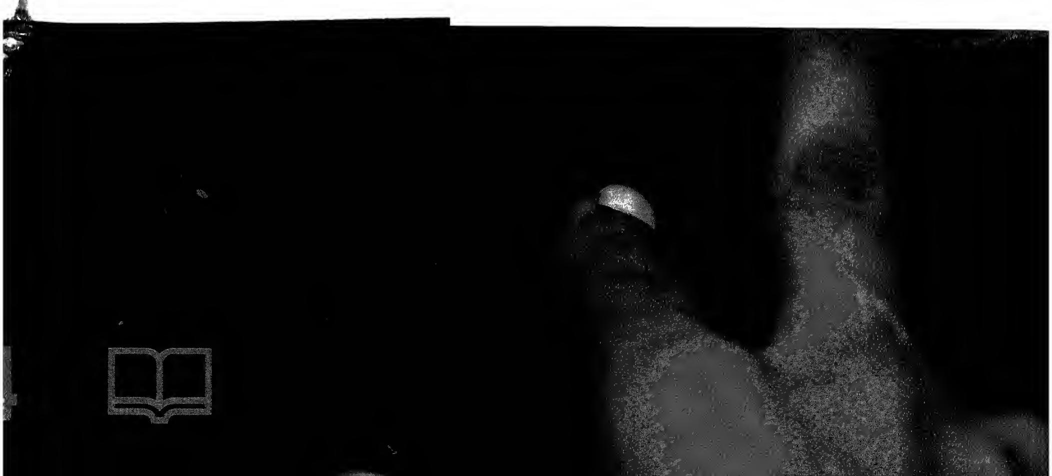


Fig. 160.

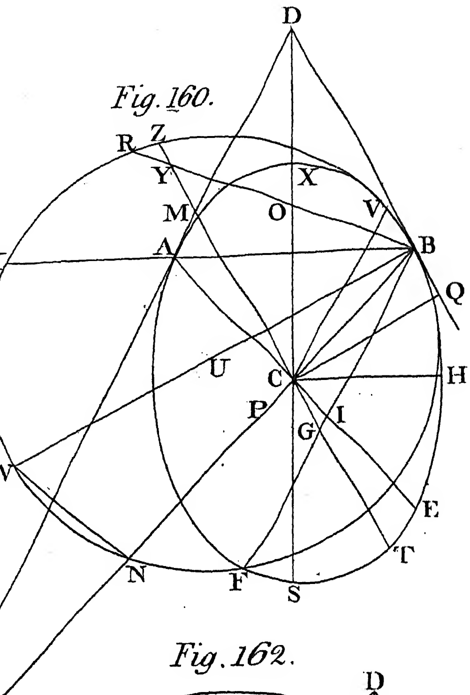
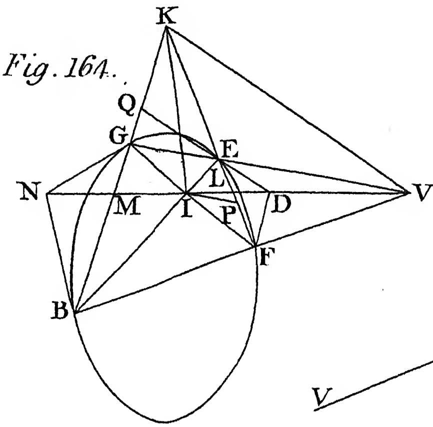


Fig. 164.



Fig

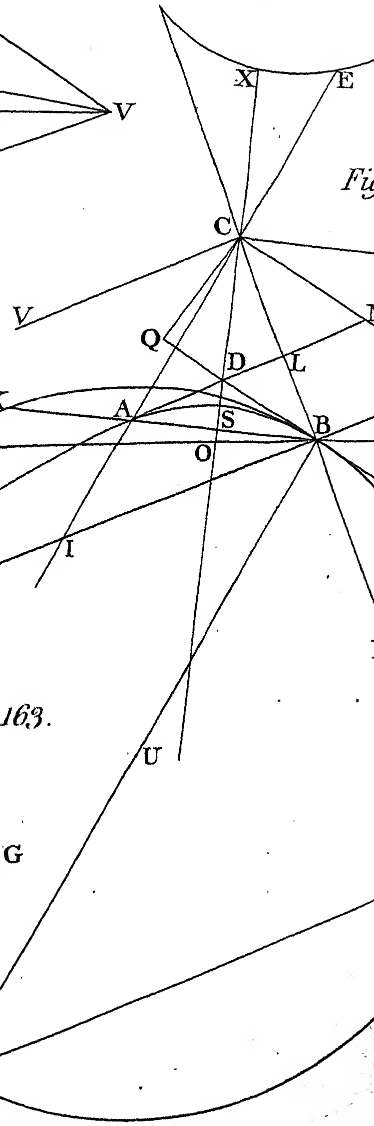


Fig. 162.

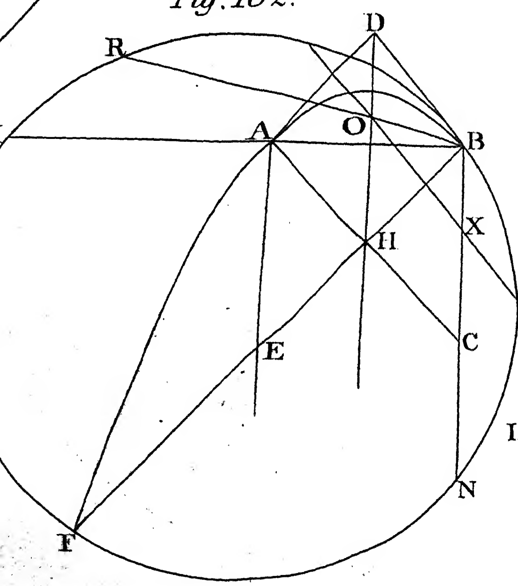
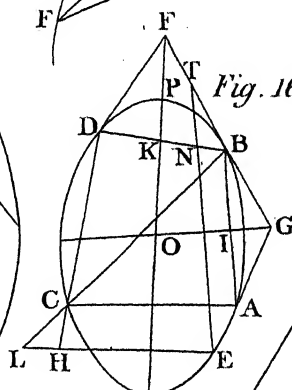


Fig. 163.



that is, by Prop. XIII. Book I. in the ratio compounded of the ratio of the rectangle B G D to the rectangle A G C, and of the ratio of the rectangle A T C to the rectangle B F D. The square of E T therefore (II. v.) is to the square of E F in the ratio compounded of the ratio of the rectangle B G D to the rectangle A G C, and of the ratio of the rectangle A T C to the rectangle B F D. BOOK IV.

Secondly, let the section be an ellipse, of which O is the center, and, the rest being as above, let O a be the semidiameter parallel to B D, O d the semidiameter parallel F T, and O b the semidiameter parallel to A C. Let it be $VX^2 : AT \times TC :: Oa^2 : Ob^2$; and then, as by Prop. V. Book II. $AT \times TC : ET^2 :: Ob^2 : Od^2$, we have Fig. 177.

$$VX^2 : AT \times TC : ET^2$$
$$Oa^2 : Ob^2 : Od^2.$$

Consequently (22. v.) $VX^2 : ET^2 :: Oa^2 : Od^2$; and therefore by Prop. V. Book II. (and 11. v.) $VX^2 : ET^2 :: BF \times FD : EF^2$, and by alternation $ET^2 : EF^2 :: VX^2 : BF \times FD$. Also, by the above, and Prop. V. Book II. (and 11. v.) $VX^2 : AT \times TC : BG \times GD : AG \times GC$. Again, the square of V X is to the rectangle under B F, F D in the ratio compounded of the ratio of the square of V X to the rectangle under A T, T C, that is, by the above, of the rectangle under B G, G D to the rectangle under A G, G C, and of the ratio of the rectangle under A T, T C to the rectangle under B F, F D. The square of E T therefore (11. v.) is to the square of E F in the ratio compounded of the ratio of the rectangle under B G, G D to the rectangle under A G, G C and of the ratio of the rectangle under A T, T C to the rectangle under B F, F D.

Cor. 1. The next remaining as above, if the straight line T F meet in the point H the straight line H L, Fig. 176 177

which

BOOK IV.

which touches the fection in the point P and meets T C in L, it may be demonftrated in the fame manner that the fquare of E T is to the fquare of E H in the ratio compounded of the ratio of the fquare of P L to the rectangle under A L, L C, and of the ratio of the rectangle under A T, T C to the fquare of H P.

Cor. 2. If the fquares of the ftraight lines M N, R S, T Q, Y Z be equal to the rectangles B G × G D, A G × G C, A T × T C, B F × F D, each to each; then E T will be to E F as the rectangle under M N, T Q to the rectangle under Y Z, R S. For, by the above, the rectangle under A G, G C is to the rectangle under B G, G D as the rectangle under A T, T C to the fquare of V X; and therefore as the fquare of R S is to the fquare of M N, fo is the fquare of Q T to the fquare of V X. Confequently (22. vi.) as R S is to M N, fo is T Q to V X, and therefore (16. vi.) the rectangle under R S, V X is equal to the rectangle under M N, T Q. Again, from the above, as the fquare of E T is to the fqnare of E F, fo is the fquare of V X to the rectangle under B F, F D, or the fquare of Y Z; and therefore, as E T to E F fo is V X to Y Z, that is (1. vi.) the rectangle under V X, R S to the rectangle under Y Z, R S. Confequently, on account of the equal rectangles, E T is to E F as the rectangle under M N, T Q to the rectangle under Y Z, R S.

Cor. 3. Hence if any two ftraight lines A C, B D cut a conic fection in A, C and B, D, and meet one another in G, and meet in T and F the ftraight line T F, which touches the fection, the point of contact may be found. For let E be fuppofed to be the point of contact, and as above E T is to E F as the rectangle under a mean proportional between B G, G D and a mean proportional between A T, T C to a rectangle under a mean proportional between B F, F D and a mean

mean proportional between A G, G C. In the ſame way the point E may be found, if the tangent T F meet in H the ſtraight line H L, which touches the ſection in P, and meets in L the ſtraight line A C, which cuts the ſection in A, C, and meets T F in T. BOOK IV.

PROP. XXI. PROB. II.

Four points in the curve of a conic ſection, and a ſtraight line touching the ſection, being given, let it be required to deſcribe the ſection.

Let the four points A, B, D, C in the curve of a conic ſection, and the ſtraight line T F touching the ſection, be given in poſition; it is required to deſcribe the ſection. Fig. 176. 177. 178. 179.

Caſe I. Let the tangent T F paſs through the point B, and draw A B, B D, D C, C A. Firſt, let the oppoſite ſides A C, B D of the trapezium be parallel, and let K R be drawn biſecting B D in K, and A C in R. Then K R is a diameter, by Cor. 2. Prop. II. Book III. and if it is parallel to the tangent F T, it will be the conjugate diameter to B D, by Cor. 2. Prop. IV. Book II. and the point K will be the center of the ſection. Conſequently, if from the point A a ſtraight line be drawn ordinately applied to the diameter B D, the magnitude of the diameter K R will be determined, by art. 10. and the ſection may then be deſcribed, by art. 8. But if the diameter K R be not parallel to the tangent F T, let them meet in F. Draw F D, and it will touch the ſection, by Cor. 2. Prop. VI. Book III. Through C draw the ſtraight line C I parallel to the tangent F T, and let it meet the tangent F D in E, and B D in G. Let the ſegment E I be ſo taken in C G, that C E may be to E G as E G to E I; and when the point G is without the ſection, let the point C be between E and I. Fig. 178.

Then

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Then the point I will be in the curve of the section, by Prop. XVII. Book I. and the five points A, B, D, C, I being in the curve, the section may be described as in Prop. XIX.

Fig. 179. Secondly, let the opposite sides of the trapezium be not parallel, and let A B, C D meet in L. Draw A D, C B, and let them meet in P. Through the points P, L draw the straight line P L, and let it meet the tangent T F in F, and draw F D. Then F D will touch the section, by Prop. XVIII. and, as above, a fifth point I may be found in the curve.

Fig. 176. 177. Case 2. If the tangent T F do not pass through a given point, the point of contact may be found, by Cor. 3. Prop. XX. and the section may be described as in Prop. XIX.

PROP. XXII. PROB. III.

Three points being given in the curve of a conic section, and two straight lines touching it being given in position, let it be required to describe the section.

Case 1. If each of the two tangents pass through one of the given points, the points of contact will be given, and a straight line drawn through the third given point in the curve, and parallel to the straight line joining the points of contact, will meet the tangents. The segments of this straight line between the given point in the curve and the tangents will therefore be given; and if they are unequal, the line will be a secant, and the other point in which it cuts the curve may readily be obtained, according to Prop. IV. The solution may then be completed, by Prop. XXI. But if the segments are equal, the line will be a tangent, and in this case call it the third tangent, and draw a straight line from its point of contact to the point of

con-

contact of the firſt tangent. Draw a ſtraight line from the point in which the firſt and third tangents meet to the point of contact of the ſecond tangent. Then, by Prop. XVI. this ſtraight line will be harmonically divided in the point in which the firſt and third tangents meet, the point in which it meets the ſtraight line joining their points of contact, the point of contact of the ſecond tangent, and the other point in which it meets the curve. This other point may therefore be found by the ſecond and third Corollaries in page 155, and the ſolution may be completed as above.

Caſe 2. Let the firſt tangent paſs through one of the given points, but let the ſecond tangent not paſs through either of the other two; and, firſt, let the two tangents be parallel. Through the two other points draw a ſtraight line, and if it be parallel to the tangents, a ſtraight line drawn from the point of contact of the firſt tangent and biſecting this ſecant will meet the ſecond tangent in its point of contact, as is evident from Prop. I. Book II. But if the ſtraight line paſſing through the other two given points be not parallel to the tangents, it will meet them, and then, by Prop. XIII. Book I. the rectangle under its ſegments between the points and the firſt tangent will be to the ſquare of the firſt tangent, as the rectangle under its ſegments between the given points and the ſecond tangent to the ſquare of the ſecond tangent. The point of contact of the ſecond tangent will therefore be obtained. Secondly, let D G be the firſt tangent paſſing through B one of the given points, and let A, C be the other two; and let D K the other tangent not be parallel to D G, but let them meet in D. Draw A C, and let it meet D K in K. Then if A C be parallel to one of the tangents, as in Fig. 180. let K E be a mean proportional between C K, K A, the point E being in K C; and Fig. 180. 181.

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and a ſtraight line drawn through B and E will meet D K in its point of contact, by Prop. XVII. Book I. But if the ſtraight line A C be not parallel to either tangent, as in Fig. 181. let it meet D G in G, and D F in K; and let G K be ſo divided, that the rectangle under A K, K C may be to the rectangle under C G, G A as the ſquare of K E to the ſquare of E G. Draw B E, and it will meet the tangent D F in the point of contact, by Cor. 2. Prop. XVII. Book I.

Fig. 182. Caſe 3. Let neither of the tangents K F or D G paſs through a given point, and let A, B, C be the given points. Draw A C, and let it meet the tangent D G in D, and the tangent K F in K. Let K D be ſo divided in E, that the rectangle under A K, K C may be to the rectangle under C D, D A as the ſquare of K E to the ſquare of E D, and the ſtraight line joining the points of contact will paſs through E, by Cor. 2. Prop. XVII. Book I. Again, draw C B, and let it meet the tangent D G in L, and the tangent K F in I. Let L I be ſo divided in M, that the rectangle under C L, L B may be to the rectangle under B I, I C as the ſquare of L M to the ſquare of M I, and the ſtraight line joining the points of contact will paſs through M, by the ſame as above. Conſequently the ſtraight line M E will meet the tangents in the points of contact.

In every caſe, therefore, a ſection may be deſcribed, by Prop. XIX. or Prop. XXI.

Cor. If two tangents be given in poſition, and alſo two points in the curve of a conic ſection, but without the tangents, the point may be found in which the ſecant, paſſing through the given points, meets the ſtraight line joining the points of contact, provided the ſecant be not parallel to the ſtraight line joining the points of contact. For if the tangents and ſecant be parallel to one another, the point in which the ſecant is

is bisected will be the point required, as stated in the second case. In other cases the Cor. is evident from the above; for the tangents D G, D K being given in position, and the secant passing through the given points A, C, as in Fig. 180, 181, 182. the point E was ascertained.

PROP. XXIII.

If three straight lines touch a conic section, a straight line drawn through the point in which the first and second tangents meet one another will be harmonically divided in this point of concourse, in the point in which it meets the third tangent, and in the points in which it meets straight lines drawn from the first and second points of contact through the third point of contact.

Let the straight lines E F, E G, G H touch the conic section in the points F, C, H, the straight line E N, drawn through the point E, in which the first tangent E F and the second E G meet one another, is harmonically divided in E, in the point R in which it meets the third tangent G H, and in the points P, N in which it meets the straight lines F H, C H, drawn from the first and second points of contact F, C through the third point of contact H. Fig. 189.

For let E N meet the curve of the section in the points S, I; and then, by Cor. 2. Prop. XVII. Book I, $SE \times EI : SR \times SI :: EP^2 : PR^2 :: EN^2 : NR^2$. Consequently (22. i.) $EP : PR :: EN : NR$.

PROP. XXIV. PROB. IV.

Two points being given in the curve of a conic section, and three straight lines being given in position and touching the curve, let it be required to describe the section.

The

BOOK IV.

The two points A, B being given in the curve of a conic section, and the three straight lines C D, E F, G H being given in position and touching the section, let it be required to describe the section.

Fig. 183. Case 1. Let C D, G H two of the given tangents pass through the two given points A, B, and let them meet the other given tangent E F in D and G. If the tangent E F be parallel to A B, the straight line joining the points of contact, and G D be bisected in E, the point E will be that in which G D touches the section, by Prop. IV. But if G D, A B be not parallel, let them meet in F, and let G D be so divided that D F may be to F G as D E to E G, and E will be the point in which G D touches the section, by Prop. XV. Three points will therefore be obtained in the curve, and consequently the solution may be completed, by Prop. XXII.

Fig. 184. 185. Case 2. Let G H one of the given tangents pass through B one of the given points, and meet the given tangent E F in G. Let the straight line A B be drawn, and let it meet the given tangent C D in D; and if A B be parallel to E F, let the segment D L be taken in B D a mean proportional between B D, D A, and the straight line joining the points in which E F, C D touch the section, will pass through the point L, by Prop. XVII. Book I. Draw G L, and let it meet the tangent C D in K; and if the tangents G H, C D be parallel, let L G be to L K as B G to K C, and C will be the point in which C D touches the section. For let G K meet the curve of the section in P and N, and the rectangle under P G, G N is to the rectangle under N K, K P as the square of G L to the square of L K, by Prop. XVII. Book I. and the square of B G is to the square of C D, the segment between the point K and the point of contact, in the same ratio, by Prop. XIII. Book I. If the straight line A B be not parallel to the tangent E F, the point

point L may be found by Cor. 2. Prop. XVII. Book I. and the tangents G H, C D being parallel, the point of contact C may be found as above. BOOK IV.

But if the tangent G H meet the tangent E F in G, and the tangent C D in H, let G H be so divided in N, that H B may be to B G as H N to G N, and the straight line joining the points of contact of the tangents E F, C D will pass through N, by Prop. XV. Again, let the straight line A B meet the tangent E F in P, and the tangent C D in D, and as the rectangle under A P, P B is to the rectangle under B D, D A, so let the square of P L be to the square of D L, and the straight line joining the points of contact of the tangents E F, C D will pass through L, by Cor. 2. Prop. XVII. Book I. Let the straight line L N therefore be drawn, and it will meet the tangents E F, C D in the points of contact. If the straight line A B be parallel to the tangent F G, the point L may be found, by Prop. XVII. Book I. as above. Fig. 185.

Case 3. Let the points A, B be without any one of the tangents, and let the straight lines A B, E F, G H be parallel. Let the tangent C D meet the tangent G H in G, and the tangent E F in E; and let the straight line A B meet the tangent C D in D. In A D let the segment D L be taken a mean proportional between A D, D B, and the straight line joining the points of contact of the tangents C D, E F will pass through the point L, by Prop. XVII. Book I. Let the straight line G L N be drawn; let it meet the tangent E F in K, and let G L be to L K as G N to N K, and the straight line joining the points of contact of the tangents G H, E F will pass through N, by Prop. XXIII. Again, let A B be bisected in P, and the straight line joining the points of contact of the tangents G H, E F will pass through P, by Prop. I. Book II. Consequently if the straight line Fig. 186.

BOOK IV.

N P be drawn, it will meet the tangents in the points of contact.

Fig. 187. Case 4. Let the tangent C D meet the tangent G H in G and E F in E, and let the straight line A B be parallel to the tangent C D, and meet the tangent G H in K, and the tangent E F in M. In the straight line K M let the segment M P be taken a mean proportional between A M, M B; and let the segment K L be a mean proportional between B K, K A; and the straight line joining the points of contact of the tangents C D, E F will pass through the point P, but the straight line joining the points of contact of the tangents C D, G H will pass through the point L, by Prop. XVII. Book I. Let the straight line G P be drawn, and meet the tangent E F in I, and as G P to P I so let G N be to N I. Again, let L E be drawn, and let it meet the tangent G H in Q, and let Q L be to L E as Q R to R E. Then, by Prop. XXIII. the straight line N R passes through the points of contact of the tangents E F, G H.

Fig. 188. Case 5. Let the straight line A B be not parallel to any one of the tangents G H, D C, E F, and let it meet the tangents D C, E F in E, the point in which they meet one another. Let the straight line A B be divided in the point L, so that B E may be to E A as B L to L A, and the straight line joining the points of contact of the tangents D C, E F will pass through the point L, by Prop. XVI. Let the straight line G L be drawn, and meet the tangent E F in K, and in the straight line G L let the segment G N be so taken, that G L may be to L K as G N to N K; and, by Prop. XXIII. the straight line joining the points of contact of the tangents E F, G H passes through the point N. Again, let the straight line A B meet the tangent G H in M, and in A B let the segment A R be so taken that the rectangle A E B may be to the rectangle A M B as the square of E R to the square

quare of R M, and the ſtraight line joining the points f contact of the tangents E F, G H will paſs through he point R, by Prop. XVII. Book I. Conſequently he ſtraight line N R will meet the tangents E F, G H n the points of contact.

Caſe 6. Let the tangent C D meet the tangent G H n G, and the tangent E F in E. Let the ſtraight line A B be drawn, and let it meet the tangent G H in M, he tangent E F in K, and the tangent C D in D. Let he ſtraight line A B be ſo divided in L and N, that the rectangle under A K, K B may be to the rectangle under B M, M A as the ſquare of K L to the ſquare of M L; and as the rectangle under A D, D B to the rectangle under B M, M A, ſo let the ſquare of D N be to the ſquare of N M. Then the ſtraight line joining the points of contact of the tangents E F, G H will paſs through the point L, but the ſtraight line joining the points of contact of the tangents C D, G H will paſs through N, by Prop. XVII. Book I. Let the ſtraight line N E be drawn, and meet the tangent G H in R; and let N E be ſo divided in P, that E N may be to N R as E P to P R; and, by Prop. XXIII. the ſtraight line joining the points of contact of the tangents E F, G H paſſes through P. Conſequently if the ſtraight line L P be drawn, it will meet the tangents E F, G H in the points of contact. Fig. 189.

In every caſe therefore the ſection may be deſcribed by Prop. XIX. as five points may be eaſily found.

PROP. XXV.

If the four ſtraight lines A E, E G, G H, H D *touch a conic ſection in* A, B, C, D, *and meet one another in* E, G, F, H; *and if the ſtraight lines* A C, B D *be drawn joining the oppoſite points* A, C *and* B, D *and meeting one another; the ſtraight line* F G *drawn through the oppo-* Fig. 174.

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ſite points in which the tangents meet one another will paſs through the point in which the ſtraight lines A C, B D *meet one another.*

For let the ſtraight line F G meet the ſection or oppoſite ſections in N, M and the ſtraight line A C in I, and, by Cor. 2. Prop. XVII. Book I. the rectangle under N G, G M is to the rectangle under N F, F M as the ſquare of G I to the ſquare of F I. But, if it be poſſible, let the ſtraight line F G meet the ſtraight line B D in P, and, as above, the rectangle under N G, G M is to the rectangle under N F, F M as the ſquare of P G to the ſquare of F P. Conſequently (11. v.) as the ſquare of G I to the ſquare of F I, ſo is the ſquare of P G to the ſquare of F P; and therefore (22. vi.) as G I to F I ſo is P G to F P. Conſequently (18. v.) as G I to F G ſo is G P to F G, and therefore (14. v.) the ſtraight lines G I, G P are equal; which is abſurd.

The reſt remaining, if the ſtraight line F G be a diameter of a parabola, or parallel to an aſymptote of an hyperbola, the ſquare of N I, and alſo the ſquare of N P, will be equal to the rectangle under G N, N F, by Prop. XXIII. Book III. and therefore G I, G P are equal; which is abſurd.

In any caſe therefore the ſtraight lines A C, B D, F G meet one another in I, and if the ſtraight line E H be drawn joining the remaining oppoſite points, in which the tangents meet one another, it will paſs through the point I, for the ſame reaſons as above.

Cor. Hence, if four ſtraight lines A E, E G, G H, H D touch a conic ſection and meet one other in E, G, F, H, and if the ſtraight lines F G, E H joining the oppoſite points meet one another in I; the ſtraight lines A C, B D joining the oppoſite points of contact will paſs through the point I.

PROP.

PROP. XXVI. PROB. V.

The point A *being given in the curve of a conic ſection, and the four ſtraight lines* E F, E G, G H, H D *touching the ſection being given in poſition, let it be required to deſcribe the ſection.* Fig. 174. 175.

Let the tangents meet one another in E, F, G, H, and let the ſtraight lines F G, E H be drawn, joining the oppoſite points F, G and E, H, and meeting one another in I. Let the ſtraight line A I be drawn, and let it meet the oppoſite tangent G H in C, and if the point A be in one of the tangents, the ſtraight line G H will touch the ſection in C, as is evident from Prop. XXV. But if the point A be not in one of the tangents, let A C meet E F in K, and let the ſtraight line K C be ſo divided in R that the ſquare of K I may be to the ſquare of I C as the rectangle A K R to the rectangle R C A; and as by the laſt Prop. the ſtraight line joining the points of contact of E F, G H paſſes through I, the point R will be in the curve of the ſection, by Cor. 2. Prop. XVII. Book I. Conſequently in any caſe the ſection may be deſcribed, by Prop. XXIV.

PROP. XXVII. PROB. VI.

Five ſtraight lines being given in poſition and touching a conic ſection, let it be required to find the points in which they touch the ſection.

Let the ſtraight lines A B, B C, C D, D E, E A touch a conic ſection, and let it be required to find the points of contact in them. Fig. 190.

Let A B C D E A be the quinquelateral figure contained by the tangents, and let A B be called the firſt ſide, B C the ſecond, &c. and let F B C D be the quadrilateral figure contained under the four firſt ſides, and draw the diagonals B D, F C meeting one another in M.

The

BOOK IV.

The firſt ſide A B of the quinquelateral figure being now omitted, let I C D E be a quadrilateral contained by the others, and let I D, C E the diagonals be drawn meeting one another in N. Then M N being drawn, it will paſs through the points in which the ſecond ſide B C and the fourth D E touch the ſection, by Cor. Prop. XXV. In the ſame manner the points may be found in which A B, C D, A E touch the ſection, and therefore the ſection may be deſcribed, by Prop. XIX.

PROP. XXVIII.

Fig. 191. *Let* E D *be an equilateral hyperbola, of which* A F, A C *are the aſymptotes, and let it cut in the point* D *the curve of the parabola* A D, *of which* A F *is the axis, and the ſegment* A F *equal to the parameter of the axis; let there be drawn to the curve of the hyperbola the ſtraight line* F E *parallel to the aſymptote* A C, *and from the point* D, *in which the curves of the hyperbola and parabola cut one another, let there be drawn to the aſymptote* A F *the ſtraight line* D B *parallel to the aſymptote* A C; *then will the ſtraight lines* B D, A B *be two mean proportionals between* A F, F E.

For as E D is an equilateral hyperbola, the angle A F E is a right one, by Prop. XVI. Book III. (and 29. i.) The ſtraight line D B is therefore an ordinate to the axis of the parabola, and, by Prop. III. Book III. (and 17. vi.) A F : B D : : B D : A B. Again, by Cor. 2. Prop. XVII. Book III. A F : A B : : B D : F E, and therefore by alternation A F : B D : : A B : F E. Conſequently (11. v.) A F : B D : : B D : A B, and B D : A B : : A B : F E.

Cor. Hence if two ſtraight lines as A F, F E be given, two mean proportionals may be found between them. For let the two ſtraight lines A F, F E be at right an-

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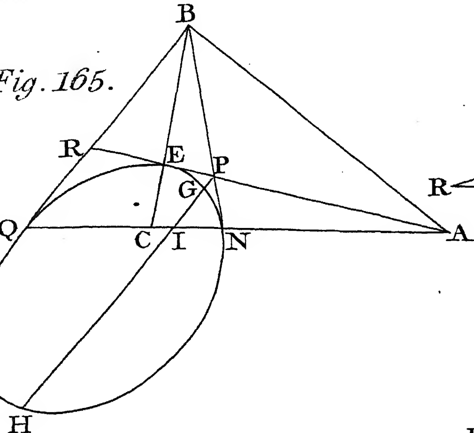


Fig. 165.

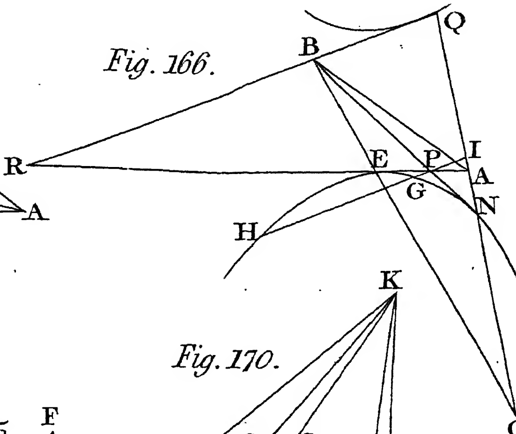


Fig. 166.

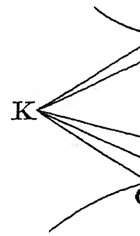


Fig. 17

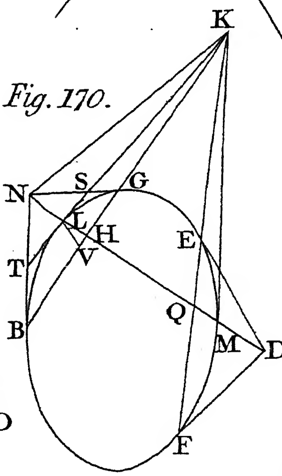


Fig. 170.

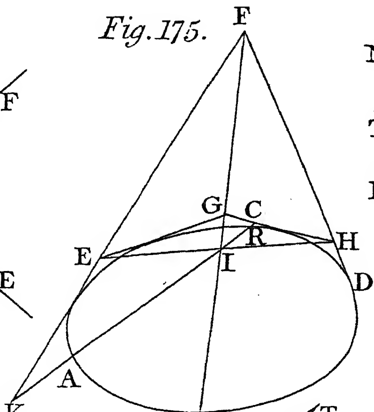


Fig. 175.

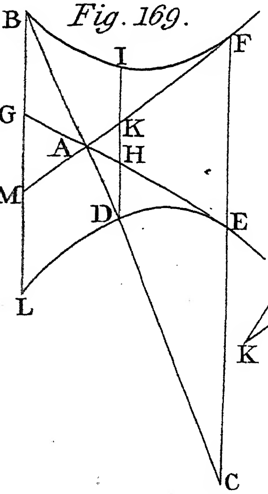
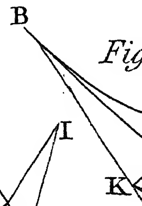


Fig. 169.



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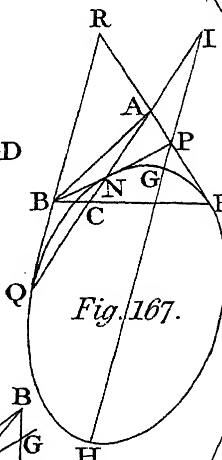


Fig. 167.

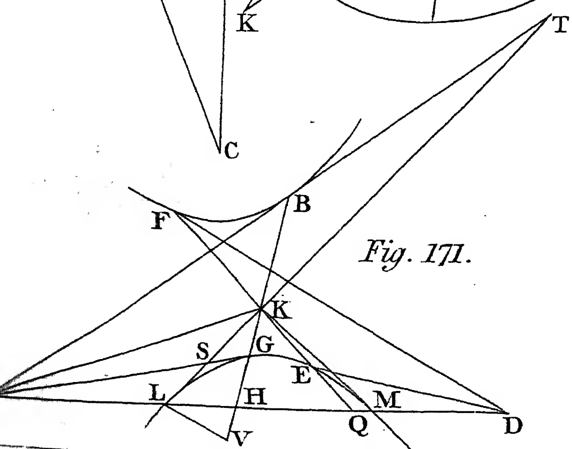


Fig. 171.

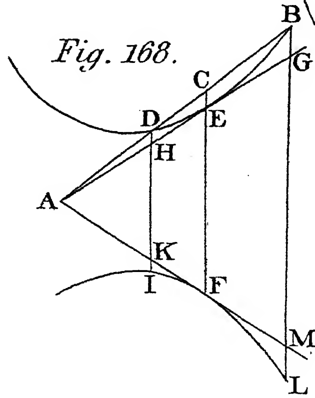
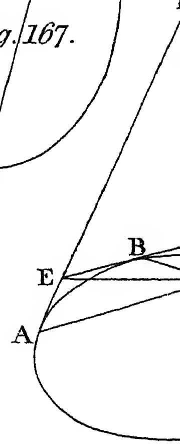


Fig. 168.



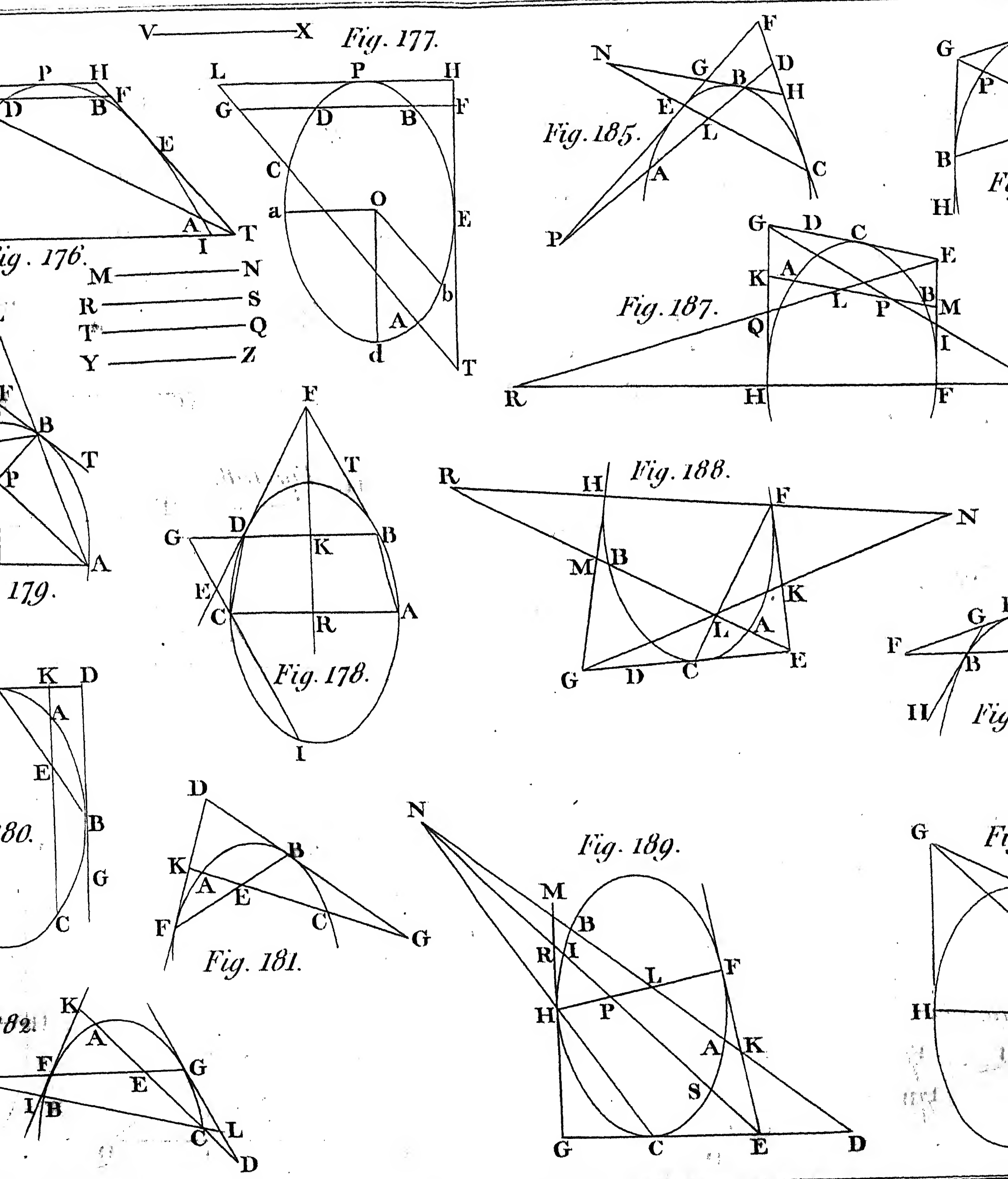


Fig. 176.

Fig. 177.

Fig. 178.

Fig. 179.

Fig. 180.

Fig. 181.

Fig. 182.

Fig. 185.

Fig. 187.

Fig. 188.

Fig. 189.

gles to one another, and let the parallelogram A F E G be completed. Let the parabola A D be defcribed, of which A F is the axis, and the fegment A F equal to its parameter. Again, let an equilateral hyperbola be defcribed through the point E, of which A F, A G are the afymptotes, and let its curve cut the curve of the parabola in D. Let D B be drawn to A F and parallel to A G, and let D C be drawn to A G parallel to A F. Then the ftraight lines B D, A B will be two mean proportionals between A F, F E.

PROP. XXIX.

Fig. 192.

Let A E *be a parabola, of which* A D *is the axis, and* A B *a fegment in it equal to half its parameter; let the ftraight line* B G *be perpendicular to the axis, and draw* A G*; with the center* G *and diftance* G A *defcribe the circle* A C E *cutting the axis in the point* C *and the curve of the parabola in* E*, and let* E D *be drawn an ordinate to the axis; the ftraight lines* E D, A D *will be two mean proportionals between* A C *and a ftraight line equal to the double of* G B.

For, by the conftruction, (and 3. iii.) the ftraight line A C is equal to the parameter of the axis, and therefore, by Prop. III. Book III. the fquare of D E is equal to the rectangle under A C, A D. Let D E meet the circumference of the circle again in F, and let the fegment F H be equal to the fegment D E. Then the rectangle E D F will be equal to the rectangle A D C, (35. iii.) and therefore the fquare of D E together with the rectangle E D F are equal to the rectangles D A C, A D C together, that is, (2. ii.) to the fquare of A D. But the fquare of D E together with the rectangle E D F is equal to the rectangle under D E, and a ftraight line equal to the fum of E D, D F (1. ii.); and therefore

BOOK IV.

fore the ſquare of D E together with the rectangle E D F is equal to the rectangle under D E, H D. Conſequently (17. vi.) D E : A D : : A D : H D; and by the above A C : D E : : D E : A D. But D H is double of G B; for let G I be drawn parallel to A D, and let it meet D H in I. Then G B, I D (34. i.) are equal to one another, as are alſo (3. iii.) E I, I F to one another, and therefore H I is equal to I D. The Propoſition is therefore evident.

Cor. Hence, by means of a parabola and a circle, a method is evident of finding two mean proportionals between two given ſtraight lines.

PROP. XXX.

Fig. 193. *From any point* B *in the curve of the equilateral hyperbola* B E *let the ſtraight lines* B A, B D *be drawn to the aſymptotes* C A, C D, *and let* B A *be parallel to* C D *and* B D *parallel to* C A, *and let* A D *the diameter of the parallelogram be drawn; with the center* B *and a diſtance equal to the double of* A D *let a circle be deſcribed, and let it meet the curve of the hyperbola in* E; *from* E *draw* E F *to the aſymptote* C D *and parallel to* C A; *then,* A F *being drawn, the angle* B A F *will be a third part of the angle* B A D.

For let A F meet B D in G. Biſect D F in K, and draw K I parallel to B D, and let it meet A F in the point I. Draw D I. Then as the hyperbola is equilateral, the angle A C D is a right one, and therefore (29. i.) each of the angles F K I, D K I is a right one, and (4. i.) F I, I D are equal. But, on account of the equals F K, K D and the parallels K I, D G, F I is equal to I G. Again, (15. and 29. i.) the triangles A B G, F C A are equiangular, and therefore (4. vi.) A B : B G : : C F : C A, and (16. vi.) the rectangle under B G, C F is equal to

to the rectangle under B A, A C. But, by Prop. XVII. Book III. the rectangle under B A, A C is equal to the rectangle under E F, C F, and therefore the rectangle under B G, C F is equal to the rectangle under E F, C F. Consequently B G is equal to E F, and therefore (33. i.) B E is equal to G F; and therefore, by the construction, F I, I D, D A are equal. The angles D F I, F D I are therefore equal to one another, as are also the angles D I A, D A I to one another. The angle D A I is therefore equal to the double (32. i.) of D F I, or of its equal the angle B A G. Consequently the angle B A G is equal to a third part of the angle B A D.

Cor. Hence, by means of an equilateral hyperbola and its asymptotes, an angle may be divided into three equal parts.

A

TREATISE

ON THE

PRIMARY PROPERTIES

OF

CONCHOIDS, THE CISSOID, THE QUADRATRIX, CYCLOIDS, THE LOGARITHMIC CURVE,

AND

THE LOGARITHMIC, ARCHIMEDEAN, AND HYPERBOLIC SPIRALS.

A

TREATISE

ON THE

PRIMARY PROPERTIES

OF

CONCHOIDS, THE CISSOID, THE QUADRATRIX, CYCLOIDS, THE LOGARITHMIC CURVE,

AND THE

OGARITHMIC, ARCHIMEDEAN, AND HYPERBOLIC SPIRALS.

SECTION I.

Of Conchoids.

DEFINITIONS.

I.

F the fixed point P be without the ſtraight line R X, and if the ſtraight line D L of indefinite length aſs through P, and D, A be two fixed points in D L; hen if the ſtraight line D L, always paſſing through P, e moved in ſuch a manner that the point A is always I T R X, and the point D deſcribe the curve D V G, the urve D V G is called a *Conchoid*. Plate XXVI. Fig. 1. 2. 3.

II.

The fixed point P is called the *Pole* or *Center* of the onchoid, and the ſtraight line T R X is called the *Directrix* of the conchoid.

SECT. I.

III.

The ſtraight line P R V perpendicular to the directrix, meeting it in R and the curve in V, is called the *Axis* of the conchoid; and the point V is called the *Vertex* of the axis.

IV.

If the directrix be between the curve and the pole, as in Fig. 1. the curve is called the *Superior Conchoid*, or the *Conchoid of Nichomedes*.

V.

If the curve be between the directrix and the pole, and the ſegment R V of the axis be leſs than the ſegment P R, as in Fig. 2. the curve is called the *Inferior Conchoid*.

VI.

If the curve and pole be on the ſame ſide of the directrix, and the ſegment V R of the axis be greater than the ſegment P R, as in Fig. 3. the curve D P V G is called the *Nodated Conchoid*.

Corollary to the preceding Definitions. In any one of the three conchoids if a ſtraight line paſs through the pole, and cut the directrix and curve, its ſegment intercepted between the directrix and the curve will be equal to the ſegment R V of the axis, between the directrix and curve. For the points D, A being fixed in the ſtraight line D L, the magnitude of the ſegment D A is the ſame in every poſition of the moving line D L; and when D L falls upon the axis the deſcribing point D coincides with V, and the point A coincides with R.

VII.

A ſtraight line drawn from any point in the conchoid perpendicular to the axis is called an *Ordinate* to the axis.

PROP.

PROP. I.

SECT. I.

The conchoid and its directrix being produced, on either side of the axis, continually approach nearer and nearer to one another, but never meet.

For, the reft remaining as in the Definitions of each of the conchoids, let the ftraight line D B be perpendicular to the directrix T X. Then P R, D B being perpendicular to the directrix T X, the triangles (15. and 29. i.) A P R, A D B are equiangular; and therefore (4. vi.) A P : P R : : A D : D B. Confequently, as by the Cor. to the Definitions A D is equal to R V, the rectangle (16. vi.) under A P, D B is equal to the rectangle under P R, R V. But D B is the diftance of the curve at the point D from the directrix; and it is evident that A P increafes as the diftance of D from the axis increafes. The diftance of the defcribing point D, therefore, from the directrix muft decreafe as D recedes from the vertex, as P R, R V are conftant; and as the rectangle under A P, D B is a conftant magnitude, the points D and B cannot coincide. Hence the Propofition is evident. Fig. 1. 2. 3.

Cor. The directrix is alfo an afymptote to the conchoid.

PROP. II.

If an ordinate be drawn from any point in either of the conchoids to the axis, a ftraight line drawn from the pole to the fame point in the curve will be a fourth proportional to the diftance of the ordinate from the directrix, the diftance of the vertex from the directrix, and the diftance of the ordinate from the pole.

The reft remaining as in the preceding Propofition, Fig. 1. 2. 3.

 and

SECT. I.

and the six first Definitions, let D E be an ordinate t the axis according to the seventh Definition, and let meet the axis in E; and then R E is to R V as P E t P D.

For B E is a parallelogram, and therefore (34. i.) R is equal to B D, and, by the Cor. to the Definitions A D is equal to R V. Consequently R E : R V : : B D A D. But the triangles (29. i.) B D A, E P D are equiangular, and therefore (4. vi.) B D : A D : : P E : P D Consequently (11. v.) R E : R V : : P E : P D.

Cor. 1. The rest remaining as above, with R as a center and R V as a distance describe V I C a quadrant of a circle, and let it cut D E, or D E produced, in I, and draw I R. Then I R is equal to R V, and therefore by the above R E : I R : : P E : P D. In the superior conchoid the angle at E is common to the two triangles R E I, P E D, and in the other two conchoids, the angle at E in the triangle R E I is equal to the angle at E in the triangle P E D, each of them being a right angle. Consequently (7. vi.) the triangles R E I, P E D are equiangular, and therefore (4. vi.) P E : E D : : R E : I E.

Cor. 2. From the preceding Cor. the equation of each of the conchoids may be easily deduced. For in each of them put R V $= a$, P R $= b$, R E $= x$, and D E $= y$.

Fig. 1. 1. In the superior conchoid (47. i.) I E $= \sqrt{a^2 - x^2}$, as I R, R V are equal; and P E $= b + x$. Consequently $b + x : y : : x : \frac{xy}{b+x} =$ I E $= \sqrt{a^2 - x^2}$; and $y = \frac{\overline{b+x} \times \sqrt{a^2 - x^2}}{x}$.

Fig. 2. 2. In the inferior conchoid I E $= \sqrt{a^2 - x^2}$, for the same reasons as above; and P E $= b - x$. Proceeding there-

therefore as in the laſt article, the equation of the inferior conchoid is $y = \frac{\overline{b-x} \times \sqrt{a^2 - x^2}}{x}$. SECT. I.

3. In the nodated conchoid IE $= \sqrt{a^2 - x^2}$, as in the other two, and PE $= b - x$; and therefore the equation of the nodated conchoid is alſo $y = \frac{\overline{b-x} \times \sqrt{a^2 - x^2}}{x}$. Fig. 3.

4. The foregoing equations being cleared of the ſurd, the equation of the ſuperior conchoid becomes $x^4 + 2\,b\,x^3 + b^2\,x^2 - a^2\,x^2 + y^2\,x^2 - 2\,a^2\,b\,x = a^2\,b^2$; and the equation of each of the other two becomes $x^4 - 2\,b\,x^3 + b^2\,x^2 - a^2\,x^2 + y^2\,x^2 + 2\,a^2\,b\,x = a^2\,b^2$.

SCHOLIUM.

Nicomedes, the inventor of the conchoïd, publiſhed an account of an inſtrument for the deſcription of the curve *, conſtructed upon the principles ſtated in the firſt Definition, of which the following is the ſubſtance.

Let TX, PR be two rulers of wood or metal, fixed at right angles to one another at R; and let them be of indefinite length, and have each a ſmooth groove to a convenient extent, as repreſented. Let DL be another ruler of wood or metal, of indefinite length, and let it alſo have a ſmooth groove of a convenient extent, as repreſented. Let P be a pin which may be fixed in the ruler PR at any requiſite diſtance from the point R. Let A be a pin which may be fixed in the ruler DL at any requiſite diſtance from the extremity D. Let the ruler DL be adapted to the other two by means of the pins A, P in ſuch a manner, that the pin A, fixed in DL, may ſlide ſmoothly in the groove in TX, and that the groove in DL may always embrace, Fig. 4.

* See the Oxford edition of Archimedes, page 147.

SECT. I. and the fix firft Definitions, let D E be an ordinate to the axis according to the feventh Definition, and let it meet the axis in E; and then R E is to R V as P E to P D.

For B E is a parallelogram, and therefore (34. i.) R E is equal to B D, and, by the Cor. to the Definitions, A D is equal to R V. Confequently R E : R V : : B D : A D. But the triangles (29. i.) B D A, E P D are equiangular, and therefore (4. vi.) B D : A D : : P E : P D. Confequently (11. v.) R E : R V : : P E : P D.

Cor. 1. The reft remaining as above, with R as a center and R V as a diftance defcribe V I C a quadrant of a circle, and let it cut D E, or D E produced, in I, and draw I R. Then I R is equal to R V, and therefore by the above R E : I R : : P E : P D. In the fuperior conchoid the angle at E is common to the two triangles R E I, P E D, and in the other two conchoids, the angle at E in the triangle R E I is equal to the angle at E in the triangle P E D, each of them being a right angle. Confequently (7. vi.) the triangles R E I, P E D are equiangular, and therefore (4. vi.) P E : E D : : R E : I E.

Cor. 2. From the preceding Cor. the equation of each of the conchoids may be eafily deduced. For in each of them put R V $= a$, P R $= b$, R E $= x$, and D E $= y$.

Fig. 1. 1. In the fuperior conchoid (47. i.) I E $= \sqrt{a^2 - x^2}$, as I R, R V are equal; and P E $= b + x$. Confequently $b + x : y : : x : \frac{xy}{b+x} =$ I E $= \sqrt{a^2 - x^2}$; and $y = \frac{\overline{b+x} \times \sqrt{a^2 - x^2}}{x}$.

Fig. 2. 2. In the inferior conchoid I E $= \sqrt{a^2 - x^2}$, for the fame reafons as above; and P E $= b - x$. Proceeding there-

therefore as in the laft article, the equation of the inferior conchoid is $y = \frac{\overline{b-x} \times \sqrt{a^2 - x^2}}{x}$. SECT. I.

3. In the nodated conchoid $IE = \sqrt{a^2 - x^2}$, as in the other two, and $PE = b - x$; and therefore the equation of the nodated conchoid is alfo $y = \frac{\overline{b-x} \times \sqrt{a^2 - x^2}}{x}$. Fig. 3.

4. The foregoing equations being cleared of the furd, the equation of the fuperior conchoid becomes $x^4 + 2bx^3 + b^2x^2 - a^2x^2 + y^2x^2 - 2a^2bx = a^2b^2$; and the equation of each of the other two becomes $x^4 - 2bx^3 + b^2x^2 - a^2x^2 + y^2x^2 + 2a^2bx = a^2b^2$.

SCHOLIUM.

Nicomedes, the inventor of the conchoid, publifhed an account of an inftrument for the defcription of the curve *, conftructed upon the principles ftated in the firft Definition, of which the following is the fubftance.

Let T X, P R be two rulers of wood or metal, fixed at right angles to one another at R; and let them be of indefinite length, and have each a fmooth groove to a convenient extent, as reprefented. Let D L be another ruler of wood or metal, of indefinite length, and let it alfo have a fmooth groove of a convenient extent, as reprefented. Let P be a pin which may be fixed in the ruler P R at any requifite diftance from the point R. Let A be a pin which may be fixed in the ruler D L at any requifite diftance from the extremity D. Let the ruler D L be adapted to the other two by means of the pins A, P in fuch a manner, that the pin A, fixed in D L, may flide fmoothly in the groove in T X, and that the groove in D L may always embrace, Fig. 4.

* See the Oxford edition of Archimedes, page 147.

SECT. I.

but ſlide ſmoothly over the pin P, fixed in P R. Then if a pencil or pen be attached to the fixed point D in D L, it will trace out the ſuperior, inferior, or nodated conchoid, according as the conditions ſtated in the fourth, fifth, or ſixth Definition, are attended to in adjuſting the inſtrument.

PROP. III. PROB. I.

Two ſtraight lines being given, let it be required to find two means in continued proportion between them by a conchoid.

Fig. 5. Let L G, L A be two given ſtraight lines; it is required to find two means in continued proportion between them.

Let A L, L G be at right angles to one another, and complete the parallelogram A L G B. Biſect A B in D, and B G in E. Draw L D, and, being produced, let it meet G B produced in H. Draw E C perpendicular to B G, and let it meet G C equal to A D, or D B, in C. Draw H C, and G F parallel to it. With C as a pole, G F as a directrix, and a diſtance equal to A D or G C between the directrix and vertex, let a conchoid be deſcribed, and let it cut H G K in K; and F K will be equal to A D or G C, by the Cor. to the Definitions. Draw K L, and, being produced, let it meet B A in M. Then will G K, M A be the two mean proportionals required.

For (6. ii.) the rectangle under B K, K G, together with the ſquare of E G, is equal to the ſquare of E K; and therefore, by adding the ſquare of E C to theſe equals, the rectangle under B K, K G, together with the ſquare of C G, (47. i.) is equal to the ſquare of C K. By ſimilar triangles M A : A B : : M L : L K : : B G : G K; and therefore (16. vi.) M A × G K is equal to A B × B G.

SECT. I.

× B G. Again, by similar triangles, L G or A B : D B :: G H : B H, and as A B is double of D B, H G is double of B G, and therefore A D × H G is equal to A B × B G. Consequently (16. vi.) M A : A D :: H G : G K :: C F : F K; and (18. v.) M D : A D :: C K : F K. But A D is equal to F K, by construction, and therefore (14. v.) M D is equal to C K, and MD^2 is equal to CK^2. By the above therefore the rectangle under B K, K G, together with the square of C G, is equal to the square of M D, which (6. ii.) is equal to the rectangle under B M, M A together with the square of A D, or its equal C G. Consequently B K × K G is equal to B M × M A, and (16. vi.) B M : B K :: G K : M A. But by similar triangles B M : B K :: G L : G K :: M A : A L. Consequently (11. v.) G L : G K :: G K : M A :: M A : A L; and therefore G K, M A are two means in continued proportion between the given straight lines A L, L G.

PROP. IV. PROB. II.

An angle being given, let it be required to divide it into three equal parts, and by means of a conchoid.

Let A C B be a given angle, it is required to trisect it, or divide it into three equal parts. Fig. 6, 7.

With C as a center, and any convenient distance C A, describe the circle A B F, and produce the diameter B F indefinitely. With A as a pole, B F as a directrix, and a distance between the directrix and vertex equal to C A, let a conchoid be described, and let it cut the circumference of the circle in E. Draw A E, and let it meet B F in D, and the angle A D C is a third part of the given angle A C B.

For C E being drawn, E D C is an isosceles triangle, and (5. i.) the angles E D C, E C D are equal, as are

 also

SECT. I.

alſo the angles C E A, C A E. In Fig. 6. the angle A E C (32. i.) therefore is double of the angle A D C, and conſequently the angle C A D is double of the angle A D C. But the angle A C B (32. i.) is equal to the angles C A D, A D C together, and therefore the angle A D C is a third part of the angle A C B. In Fig. 7. the angle E A C, (32. i.) and therefore its equal A E C, is equal to the angles E D C, A C D together; and the angle D A C is equal to the angles A E C, A C E together. Conſequently the angle D A C is equal to the angles E D C, A C D, A C E taken together; that is, the angle D A C is equal to the angles E D C, E C D together, or to the double of the angle A D C. Conſequently, as the angle A C B (32. i.) is equal to the angles D A C, A D C together, the angle A D C is a third part of the angle A C B.

SECTION

Pla

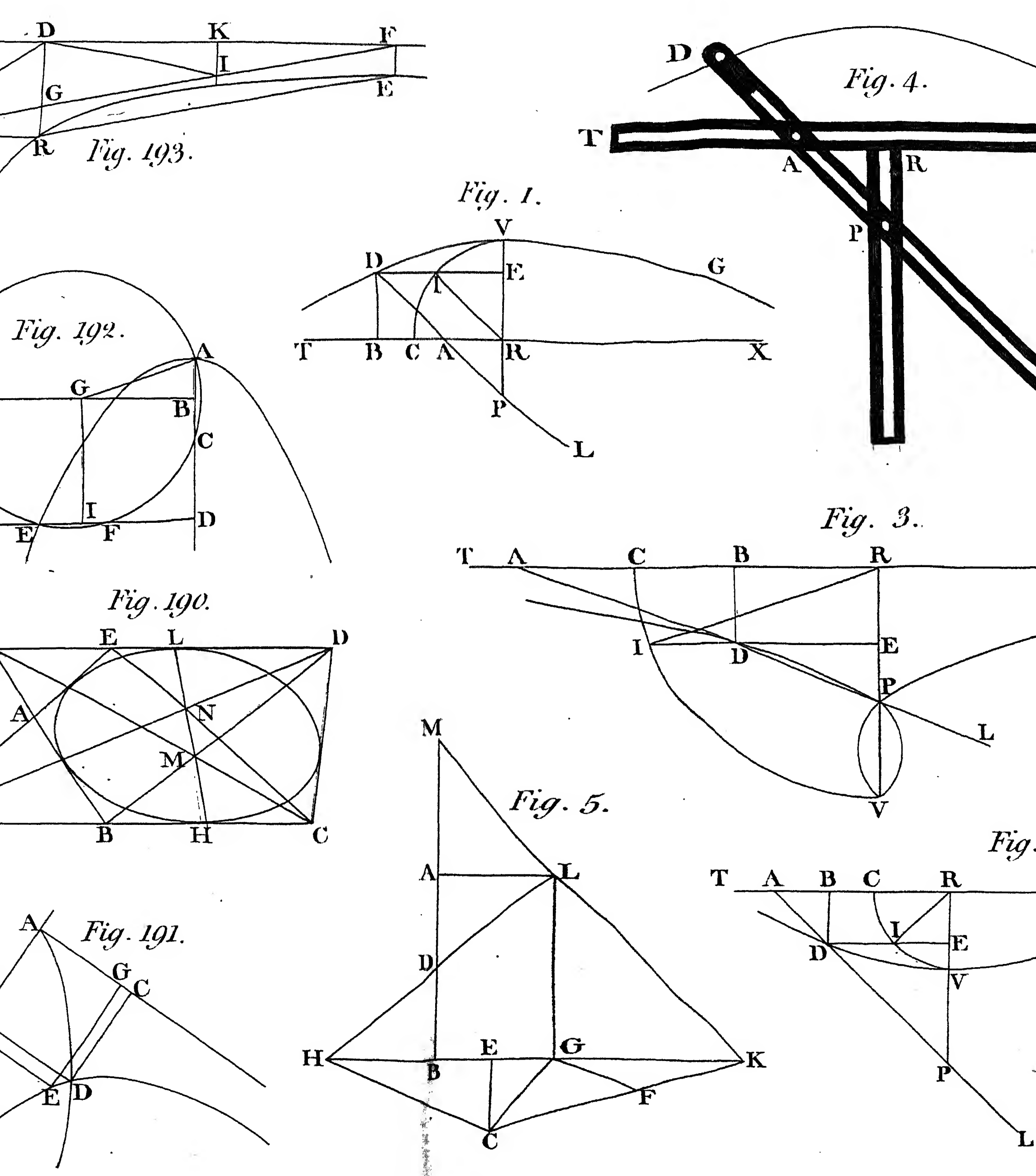


Fig. 193.

Fig. 4.

Fig. 1.

Fig. 192.

Fig. 3.

Fig. 190.

Fig. 5.

Fig.

Fig. 191.

SECTION II.

Of the Ciſſoid.

LEMMA.

Let E G F H be a circle of which C is the center, and E F, H G two diameters at right angles to one another. Let E F be produced to P, ſo that F P may be equal to C F; and A being any point in C G draw A P, and upon A P as a diameter deſcribe the ſemicircle A B P. Then if from A a ſtraight line A B, equal to C P, be inſcribed in the ſemicircle, the point in which A B is biſected will be on the ſame ſide of C P with the point A. Fig. 8.

For let K be the point in which A P is biſected, and draw K F; and draw K D perpendicular to A B. Then as A C P is a right angle, the circumference A B P (31. iii.) paſſes through C; and as A B, C P are equal, the circumference A C B (28. iii.) is equal to the circumference P B C; and therefore the circumference A C is equal to the circumference B P. The angle A P C (27. iii.) is therefore equal to the angle P A B; that is, the angle K P F is equal to the angle K A D. Again, as P K is equal to K A, and P F equal to F C, P K : K A : : P F : F C; and therefore (2. vi.) K F is parallel to C A, and (29. i.) the angle P F K is a right angle. The triangles P K F, A K D are therefore equiangular, and (26. i.) K F is equal to K D, and A D to P F. Conſequently if with K as a center and K F as a diſtance a circle be deſcribed, it will paſs through D, and C P will be a tangent to this circle (16. iii.) as it is at right angles to K F. The ſtraight line A B is alſo biſected in D, for

SECT. II.

for A D is equal to P F. Confequently the point in which A B is bifected is on the fame fide of C P with the point A.

DEFINITIONS.

I.

Fig. 9. Let E G F H be a circle, of which C is the center, and E F, H G two diameters at right angles to one another. Let E F be produced to P, fo that F P may be equal to C F the radius, and let H G be produced indefinitely on one fide towards A a point in H G. Let the ftraight line B L, of indefinite length towards L, pafs through P; and let B A, equal to E F or C P, be at right angles to B L, and let D be the point in which B A is bifected. Then if the ftraight lines B L, B A be fo moved that B L always pafs through P, and the extremity A of B A be always in H G, the point D will defcribe a curve F G D which is called a *Ciffoid*, or the *Ciffoid of Diocles*.

II.

The circle E G F H is called the *Generating Circle* of the ciffoid; and the point P is called its *Pole*.

III.

The ftraight line H G A is called the *Directrix* of the ciffoid.

IV.

The diameter E F of the generating circle is called the *Axis* of the ciffoid, and the point F is called its *Cufp*.

V.

The ftraight line E K perpendicular to the axis is called the *Afymptote* to the ciffoid.

VI.

A ftraight line drawn from any point in the ciffoid perpendicular to the axis is called an *Ordinate* to the axis,

axis, and the segment of the axis between the cusp and an ordinate is called an *Abscifs* of the axis. SECT. II.

PROP. I.

The cissoid commences at the cusp, and the curve is entirely on one side of the axis; it also passes through the point in which the directrix cuts the generating circle, and being continually produced it approaches nearer and nearer to the asymptote, but never meets it.

For, the rest remaining as in the Definitions, draw the straight line P A. Then as P C A, P B A are right angles, a semicircle described upon P A as a diameter will (31. iii.) pass through B and C; and the straight line A B, equal to C P, will be inscribed in this semicircle, in every situation of A, regulated according to the first Definition. Consequently when A coincides with C the straight line A B will coincide with C P, and the point D will coincide with F. If therefore the description of the curve be supposed to begin from this situation, the cusp F will be the point at which it commences, and as soon as A has moved from C towards G the describing point D will be removed from E F towards G, according to the Lemma prefixed to the Definitions. When the distance of A, in the directrix, from C is equal to C P or A B, then the point B will coincide with C, and the describing point D will coincide with G. Through D draw K M perpendicular to C A, and let it meet the asymptote in K. Let B P cut C A in N. Then as P C N, N B A are right angles, and (15. i.) as the angles P N C, A N B are equal, the triangles P C N, A B N are equiangular. The triangles P C N, A M D are therefore equiangular, and the angle C P N is equal to the angle M A D. But as the point A recedes from G the point N also recedes from it, and there- Fig. 9.

SECT. II. therefore the ciſſoid being continually produced the angle P N C becomes leſs and leſs, and conſequently the angle C P N, or its equal M A D, becomes greater and greater. The perpendicular D M muſt therefore continually increaſe, as the length of D A is conſtant, and conſequently D K, the diſtance of the curve from the aſymptote, muſt continually decreaſe, as K M (34.i.) is equal to E C. The point D however can never fall into E K, as D M can never become equal to D A or E C; for if it did then P B would become parallel to C A; which is impoſſible.

Cor. From the above it is evident, that an ordinate drawn from any point in the ciſſoid to the axis will cut the generating circle.

PROP. II.

An ordinate, drawn from any point in the ciſſoid to the axis, is a third proportional to its ſegment between the generating circle and the axis, and the correſponding abſciſs.

Fig. 10. 11. From D, any point in the ciſſoid F D G whoſe cuſp is F, let D R be drawn an ordinate to the axis F E, and let it cut the generating circle in T; then T R is to R F as R F to R D.

For, the reſt remaining as in the Definitions and preceding Propoſition, let D R cut the generating circle again in S, and draw C S. Then as A D is equal to C S, and (34. i.) D M equal to R C, and D M A, C R S right angles, the ſquares of M A, S R are (47. i.) equal; and conſeqnently M A, S R are equal. Again, as in the triangles A B N, P C N, A B, P C are equal, and as the angle B N A is equal to the angle C N P, and the angle A B N equal to the angle P C N, the ſide B N is equal to the ſide C N. By ſimilar triangles alſo D A : M A ::

N A

N A : B A, and M D : M A : : N B : B A. Confequently on account of the equals (24. and its firft Cor. v.) F R : S R : : C A : E F; and F R $\times$ E F is equal to S R $\times$ C A. But (3. ii.) F R $\times$ E F is equal to E R $\times$ R F together with the fquare of F R; and (35. iii.) E R $\times$ R F is equal to the fquare of S R. Confequently F R $\times$ E F is equal to the fum of the fquares of F R, S R. Again, S R $\times$ C A is equal to S R $\times$ S D, for (34. i.) R D, C M are equal, and by the above M A is equal to S R. But S R $\times$ S D (3. iii.) is equal to S R $\times$ R D together with the fquare of S R. By the above therefore the fum of the fquares of F R, S R is equal to S R $\times$ R D together with the fquare of S R. Confequently S R $\times$ R D is equal to the fquare of F R; and therefore (3. iii.) as T R is equal to S R, (17. vi.) T R : R F : : R F : R D.

SECT. II.

Cor. 1. The ftraight lines E R, R T, R F, R D are in continued proportion. For (Cor. 8. vi.) E R : R T : : R T : R F; and as above R T : R F : : R F : R D.

Cor. 2. The equation of the ciffoid is eafily deduced from the laft Corollary. For put E F $= a$, F R $= x$, and R D $= y$; and then E R $= a - x$, and R T^2 = E R $\times$ R F, or R T $= \sqrt{ax - x^2}$. Confequently $\sqrt{ax - x^2} : x : : x : \dfrac{x^2}{\sqrt{ax - x^2}} = y$, and $\dfrac{x^4}{ax - x^2} = y^2$, or $\dfrac{x^3}{a - x} = y^2$; and therefor $x^3 = ay^2 - xy^2$, which is the equation of the curve.

PROP. III.

If from the cufp of a ciffoid a ftraight line be drawn cutting the ciffoid and the generating circle, ftraight lines drawn through the points of fection, and perpendicular to the axis, will be equally diftant from the directrix.

Let F D G L be a ciffoid, of which F is the cufp, F E the axis, H C G the directrix, and E G F H the generating Fig. 12.

SECT. II.

ing circle. From F draw any ſtraight line F K cutting the ciſſoid in D and the circle in K, and let D B, K M be perpendicular to the axis; the ſtraight lines D B, K M are equally diſtant from H C G.

For let C be the center of the circle, and let the directrix cut the circle and ciſſoid in G. Let B D meet the circle in A, and let K M meet it again in N, and draw E K. Then, by Prop. II. A B : B F : : B F : B D; and (4. vi.) B F : B D : : M F : M K. Conſequently (11. v.) A B : B F : : M F : M K; and therefore (6. vi.) the triangles A F B, F K M are equiangular, and the angle A F B is equal to the angle M K F. The circumference A E therefore (26. iii.) is equal to the circumference N F, which is equal to the circumference F K. The circumference F A is therefore equal to the circumference E K, and conſequently (29. iii.) the ſtraight lines F A, E K are equal. But (Cor. 8. vi.) E M is a third proportional to E F, E K; and F B is a third proportional to E F, A F. Conſequently E M, F B are equal, and therefore C M is equal to C B.

Cor. 1. Hence it is evident that the arch G K is equal to the arch G A.

Cor. 2. If equal arches as G K, G A be ſet off in the circumference of the circle, on the oppoſite ſides of H G, and perpendiculars K M, A B be drawn to the diameter E F, a ſtraight line drawn from F to K will cut the perpendicular A B and the ciſſoid in the ſame point. For the ſame reaſons a ſtraight line drawn from F to A will cut the perpendicular M K and the ciſſoid in the ſame point. For it may be proved, as above, if the ſtraight line F A cut the generating circle in A and the ciſſoid in L, and if A B, L M be drawn perpendicular to the axis, that C B, C M are equal.

SCHO-

SCHOLIUM.

SECT. II.

Diocles, the inventor, confidered the property expreffed in the laft Corollary as the primary one of the ciffoid; and he fuppofed the defcription of the curve to be effected by means of an indefinite number of points, as D and L, obtained from equal arches as G K, G A, and the interfections of the ftraight lines F K, F A with the perpendiculars A B, M L *. Sir I. Newton firft fhewed how the ciffoid might be defcribed by continued motion according to the conditions expreffed in the firft Definition in this fection †. The method will be eafily underftood if P C A, L B A be fuppofed to be two fquares of wood or metal, P being a fixed point in P C one arm of the one, and A a fixed point in B A an arm of the other; and if the adjuftment of the inftrument and its action be fuppofed to be regulated as mentioned in the firft Definition. Fig. 9.

It is evident from the Definitions, and the Propofitions and their Corollaries, that with the fame generating circle E G F H, the fame pole P, and the fame cufp F, another ciffoid F H R may be defcribed on the oppofite fide of E P to that on which F G D is defcribed. It is alfo evident that S E K, touching the generating circle in E, is the common afymptote to thefe two ciffoids, and that E F is their common axis; and it may readily be perceived that the properties proved of the one equally apply to the other.

* See the Oxford edition of Archimedes, page 138.

† See the Appendix to the Arithmetica Univerfalis.

SECTION

SECTION III.

Of the Quadratrix.

DEFINITIONS.

I.

Fig. 13. Let A D E be a femicircle, and let C be the center of the circle, and A E a diameter. Let C D be at right angles to A E and meet the circumference in D, and let it be produced to S fo that D S may be equal to C D. Let a ftraight line C H of indefinite length revolve about C, begin its revolution from a coincidence with C E, and move towards D; and at the fame time that C H begins to revolve let a ftraight line F G move from a coincidence with C E towards S, and let F G be always parallel to A E, or perpendicular to C S. Let C H revolve and F G move with an uniform velocity, and let the arch E H paffed over by the revolution of C H be to the diftance C F moved over by the extremity of F G in the fame time, as the quadrantal arch E D to the radius C E; the curve B I D defcribed by the point I, in which C H, F G cut one another, is called a *Quadratrix*, or the *Quadratrix of Dinoftrates*.

II.

The circle A D E is called the *Generating Circle* of the quadratrix.

III.

The ftraight line C S is called the *Axis* of the quadratrix.

IV.

If B be the point at which the lines C H, F G commence

mence their interfection, the ftraight line C B is called the *Bafe* of the quadratrix. SECT. III.

V.

The ftraight line S T perpendicular to the axis is called the *Afymptote* to the quadratrix.

PROP. I.

The curve of the quadratrix paffes through the point in which the axis cuts the generating circle; and being continually produced it approaches nearer and nearer to the afymptote, but never meets it.

Part I. Every thing remaining as in the Definitions, as C H revolves and F G moves with an uniform velocity, and as the velocity with which C H revolves is to the velocity with which F G moves as the quadrantal arch E D to the radius C D, the arch E H is to C F as the quadrantal arch E D to the radius C D. When the revolving line C H therefore coincides with C D, the point F in F G will coincide with D, and I will alfo coincide with D. Confequently the curve of the quadratrix muft pafs through D. Fig. 13.

Part II. The reft remaining as above, let C L reprefent the revolving line after it has proceeded beyond C D from E, and let L M reprefent the fituation of the line moving parallel to A E at the fame time, and let C L, M L interfect one another in L, and let C L cut the generating circle in K. Then it is evident, from the firft Definition, that the point L is in the curve of the quadratrix; and, for the fame reafons as above, the arch E D K is to C M as the quadrantal arch E D to the radius C D. Hence it is evident, that, if the curve of the quadratrix be continually produced, the revolving line will cut off greater and greater arches of the generating circle, reckoning from the extremity E, and

SECT. III.

consequently the distances of the moving line parallel to A E must become greater and greater. The distance of the moving line parallel to A E from the asymptote T S must therefore continually decrease. Consequently the curve of the quadratrix must approach nearer and nearer to the asymptote upon being continually produced, but they can never meet, for if they did then the revolving line would coincide with A E, and A E, T S would meet. But this is impossible, for, by the fifth and first Definitions, (and 28. i.) they are parallel.

Cor. Any arch E H and distance C F, passed over by the revolving line C H and moving line F G in the same time, are to one another as the quadrantal arch E D to the radius C D. Also (19. v.) the arch D H is to D F as the quadrantal arch E D to the radius C D.

PROP. II.

The base of the quadratrix is a third proportional to the quadrantal arch of the generating circle and its radius.

Fig. 14. Let B I D be a quadratrix, of which C E is the base, A D E the generating circle, and C E or C D the radius of the circle, C being its center; the quadrantal arch E H D of the generating circle is to its radius C E as C E to C B.

For let C H be a position of the revolving line, and F I a corresponding position of the line which moves parallel to A E, and let them intersect one another in I, as in the first Definition, so that I may be in the curve of the quadratrix. Let C H, F I be indefinitely near to C E, so that the arch E H of the generating circle may be indefinitely small; and let H L, I K be perpendicular to C E. Then, by the Cor. to Prop. I. the arch E H : C F : : arch E H D : C E. But it is evident that F K is a parallelogram, and therefore (34. i.) C F is equal to

K I;

K I; and as the arch E H is indefinitely ſmall, it is equal to its ſine L H. Conſequently L H : K I : : arch E H D : C E; and therefore (II. v. and 4. vi.) arch E H D : C E : : C L : C K. But when the arch E H is indefinitely ſmall, and equal to L H, C L is equal to the radius, and C K becomes equal to C B. Conſequently the quadrantal arch E D : C E : : C E : C B.

Cor. 1. As by the above and inverſion C B : C E : : C E : the quadrantal arch E D, by the Cor. to Prop. I. (and II. v.) C B : C D : : F D : the arch D H. Alſo by the above, Cor. to Prop. I. (and II. v.) C B : C D : : C F : the arch E H.

Cor. 2. The equation of the quadratrix may be obtained from the above in the following manner. Put C D $= a$, C B $= b$, E H $= z$, C F $= y$; and then, as in the Propoſition $b : a : : a : \frac{a^2}{b} =$ the quadrantal arch E D. Conſequently $\frac{a^2}{b} : a : : z : \frac{bz}{a} =$ C F $= y$; and $bz = ay$, the equation of the curve.

LEMMA.

A circle is equal to a right angled triangle, which has one of the ſides round the right angle equal to the radius of the circle, and the other ſide equal to the circumference.

Let A I B D E be a circle, as in Fig. 15. of which C is the center, and C I a radius, and let K M N, as in Fig. 16. be a right angled triangle, having K M one of the ſides round the right angle at M equal to C I, and the other M N equal to the circumference of the circle; the circle A I B D E is equal to the triangle K M N.

For, if it be poſſible, let the circle be greater than the triangle, and firſt by inſcribing a ſquare in it, and afterwards by a repeated biſection of circular arches,

 let

SECT. III.

let a polygon of an even number of fides be infcribed in the circle; and let the excefs of the circle above the polygon be lefs than its excefs above the triangle. Then the polygon thus infcribed in the circle will be greater than the triangle. Let I B be a fide of this polygon, and let C L, at right angles to it, meet it in L. Then C L is lefs than K M, and as the ftraight line I B is lefs than the circular arch I B, the perimeter of the polygon is lefs than the circumference of the circle. The rectangle under C L and the perimeter of the polygon is therefore lefs than the rectangle under K M, M N. But it is evident (from 1. ii. and 34. i.) that the rectangle under C L and the perimeter of the polygon is double the area of the polygon; and the rectangle under K M, M N is double the area of the triangle K M N. The polygon is therefore lefs than the triangle K M N; and it is alfo greater; which is abfurd.

But, if it be poffible, let the circle be lefs than the triangle; and firft by defcribing a fquare about the circle, and afterwards by a repeated bifection of circular arches let a polygon be defcribed about the circle, and let the excefs of the polygon above the circle be lefs than the excefs of the triangle above the circle. And that this may be done is evident from (1. x. and 1. xii.) confidering that if B H, A H touch the circle in B and A and meet one another in H, then if C H be drawn cutting the circle in I, and F G touch it in I and meet B H in F and A H in G, and B I, I A be drawn, the triangle H I F is greater than the triangle B I F, and the triangle H I G is greater than the triangle A I G. For (18. iii.) H I F, H I G are right angles, and therefore (18. i.) F H is greater than F I, and G H greater than G I. But it is evident (from 36. iii.) that B F is equal to F I, and G I to G A, and therefore (1. vi.) the triangle H I F is greater than the triangle B I F, and the

tri-

triangle H I G is greater than the triangle A I G. Let F G be a side of the polygon described about the circle, whose excess above the circle is less than the excess of the triangle K M N above the circle, and then the polygon, of which F G is a side, is less than the triangle K M N. But as the perimeter of the circumscribed polygon is greater than the circumference of the circle, and as the rectangle under C I and the perimeter of the circumscribed polygon is equal to the double of the area of the polygon, it is evident for the same reasons as above that the circumscribed polygon is greater than the triangle K M N. The circumscribed polygon therefore is both less and greater than the triangle K M N; which is absurd. Consequently the circle A I B D E is equal to the triangle K M N. SECT. III.

Cor. 1. A circle is equal to a rectangle which has one of its sides equal to the radius of the circle, and the other side round the same angle equal to half the circumference.

Cor. 2. The diameter of one circle is to the diameter of another as the circumference of the first mentioned to the circumference of the other. For put D equal to the diameter of the one and C equal to its circumference, and put d equal to the diameter of the other and c equal to its circumference. Then, by the preceding Cor. (and 2. xii.) $D \times C : d \times c :: D^2 : d^2$, and $D \times C : D^2 :: d \times c : d^2$. Consequently (1. vi.) $C : D :: c : d$.

Cor. 3. If the two circles F H G B, K L M E have the common center C, and C A, C B be drawn cutting the outer circle in A, B, and the inner in D, E, the radius C B is to the radius C E as the arch A B to the arch D E. For let F G, H B be two diameters of the outer circle at right angles to one another, and let F G cut the inner circle in K, M, and H B cut it in L, E. Then Fig. 17.

SECT. III.

(33. vi.) the arch F B : the arch A B : : the angle F C B : the angle A C B : : the arch K E : the arch D E ; and therefore, by alternation, the arch F B : the arch K E : : the arch A B : the arch D E. But the arch F B is a fourth part of the circumference F H G B, and the arch K E is a fourth part of the circumference K L M E. Confequently, by the laft Cor. (and 15. v.) C B : C E : : the arch A B : the arch D E.

Such fectors as A C B, D C E which have the angles at their centers equal, are called *Similar Sectors.*

PROP. III.

If from any point in the quadratrix a ftraight line be drawn to the center of the generating circle, and alfo a ftraight line perpendicular to the axis, and if with the center of the generating circle, and the bafe as a diftance, a circle be defcribed, the arch of this circle intercepted between the extremity of the bafe and the ftraight line drawn from the quadratrix to the center, will be equal to the fegment of the axis between the center and perpendicular.

Fig. 19.

From any point L in the quadratrix B D L let a ftraight line L C be drawn to C the center of the generating circle A D E, and with C as a center and C B, the bafe of the quadratrix as a diftance, let a circle be defcribed; the arch B F of this circle, intercepted between B and C L, is equal to C M, the fegment of the axis C D M between C the center and L M the perpendicular.

For, by Prop. II. and inverfion, C B : C D : : C D : the quadrantal arch D E; and, by Cor. 1. to Prop. II. C D : quadrantal arch D E : : C M : the arch E K; and, by Cor. 2. to the preceding Lemma, (and 15. v.) C B : C D or C E : : the quadrantal arch B G : the quadrantal arch

rch D E. Again (33. vi.) the angle E C D : the angle C K : : the quadrantal arch E D : the arch E K ; and or the same reasons the quadrantal arch B G is to the rch B F in the same proportion. Consequently (11. v.) he quadrantal arch E D : the arch E K : : the quadrantal arch B G : the arch B F; and, by alternation, he quadrantal arch E D : the quadrantal arch B G : : he arch E K : the arch B F. By the above therefore and 11. v.) the arch B F : the arch E K : : C M : the rch E K; and (14. v.) consequently the arch B F is equal to C M. SECT. III.

Cor. The quadrantal arch B G is equal to the radius C D. For, as above, by Prop. II. C B : C D : : C D : the quadrantal arch D E; and, by Cor. 2. to the preceding Lemma, C B : C D : : the quadrantal arch B G : the quadrantal arch E D. Consequently (11. v.) C D : the quadrantal arch E D : : the quadrantal arch B G : the quadrantal arch E D; and therefore (14. v.) C D is equal to the quadrantal arch B G.

SCHOLIUM.

If the straight line C H revolve, and the straight line F G move with the same relative velocities as stated in the first Definition, but on the side of A E opposite to that before supposed, the quadratrix L D B may be extended as represented by L D B K O in Fig. 18. and if the generating circle be completed, and D C be produced to M so that C M be equal to C S, then a straight line M N at right angles to C M will also be an asymptote to the curve. It is also evident that the curve will pass through the point K in which C M cuts the generating circle. Fig. 18.

It is much to be wished that an instrument were devised for describing the quadratrix by continued motion, as the two following important Problems could

 then

SECT. III.

then be solved geometrically. In the following solutions the possibility of describing the quadratrix is necessarily supposed.

PROP. IV. PROB. I.

From a given rectilineal angle to cut off any part required, by means of the quadratrix.

Fig. 13. Let K C E be a given rectilineal angle; it is required to cut off any part from it, by means of the quadratrix.

With C as a center, and any convenient distance C E, let a circle A D E be described. With the generating circle A D E let the quadratrix B I D L be described, and let it meet C K in L. Let C D M be the axis of the quadratrix, and let L M be perpendicular to it. Let C M be so cut in F (9. vi.) that C M may be to C F as the whole given angle K C E to the part required. Draw F G parallel to C E, and let it cut the quadratrix in I. Draw C I and let it meet the circle in H; and the angle E C H will be the part required.

For, by the Cor. to Prop. I. (and 11. v.) CM : CF : : the arch K E : the arch H E; and (33. vi.) the arch K E : the arch H E : : the angle K C E : the angle H C E.

PROP. V. PROB. II.

To find a square equal to a given circle, by means of the quadratrix.

Fig. 18. Let A D E K be a given circle; it is required to find a square equal to it, by means of the quadratrix.

Let C be the center of the circle, and with A D E K as a generating circle, and its diameter D K as an axis, let the quadratrix L D B K O be described, and let C B, in the diameter A C E, be its base. Through B draw P Q parallel to D K, and let it meet

D P

D P perpendicular to D K in P, and K Q perpendicular to D K in Q. Draw C P, C Q, and let them meet R E V parallel to D K in R and V. Draw R W, V X perpendicular to D K. Then the right angled parallelogram W R V X is equal to the given circle A D E K, and a mean proportional between the ſides W R, R V is the ſide of the ſquare required.

For with C as a center, and C B as a diſtance, let the circle G B I be deſcribed, and let it meet D K in G and I. Then, by the Cor. to Prop. III. the quadrantal arch B G is equal to C D, and the quadrantal arch B I is equal to C K. Again, by the above conſtruction, D B, W E are parallelograms, and therefore (34. i.) D P is equal to C B, and W R is equal to C E. The triangles C D P, C W R are alſo equiangular, (29. i.) and therefore (4. vi.) D P : W R : : C D : C W. Conſequently, by Cor. 2. to the Lemma in this ſection, (and 15. v.) the quadrantal arch D E is equal to C W, and for the ſame reaſons the quadrantal arch K E is equal to C X; and therefore W X is equal to half the circumference of the circle A D E K. The right angled parallelogram W R V X is therefore, by Cor. 1. to the Lemma in this ſection, equal to the circle A D E K; and a mean proportional (13. vi.) being found, it will be equal to the ſide of the ſquare required.

Cor. It is evident, from the Cor. to Prop. III. and Cor. 1. to the Lemma in this Section, that the right angled parallelogram D P Q K is equal to the circle G B I.

SECTION

SECTION IV.

Of Cycloids.

DEFINITIONS.

I.

Fig. 20. If the circle A W E roll along the ſtraight line A B ſo that every part of the circumference may touch it in regular ſucceſſion, and if at the commencement of the revolution of the circle it touch A B in A, and if during the revolution the point A remain fixed in the circumference, and at the end of it meet the ſtraight line in B; the curve A V B deſcribed by the point A, during the revolution, is called a *Cycloid*.

II.

The circle A W E is called the *Generating Circle* of the cycloid.

III.

The ſtraight line A B is called the *Baſe* of the cycloid.

Cor. As every part of the circumference of the circle touches the baſe, in regular ſucceſſion, the baſe of the cycloid is equal to the circumference of the generating circle.

IV.

The ſtraight line H V, biſecting the baſe A B at right angles and meeting the curve in V is called the *Axis* of the cycloid.

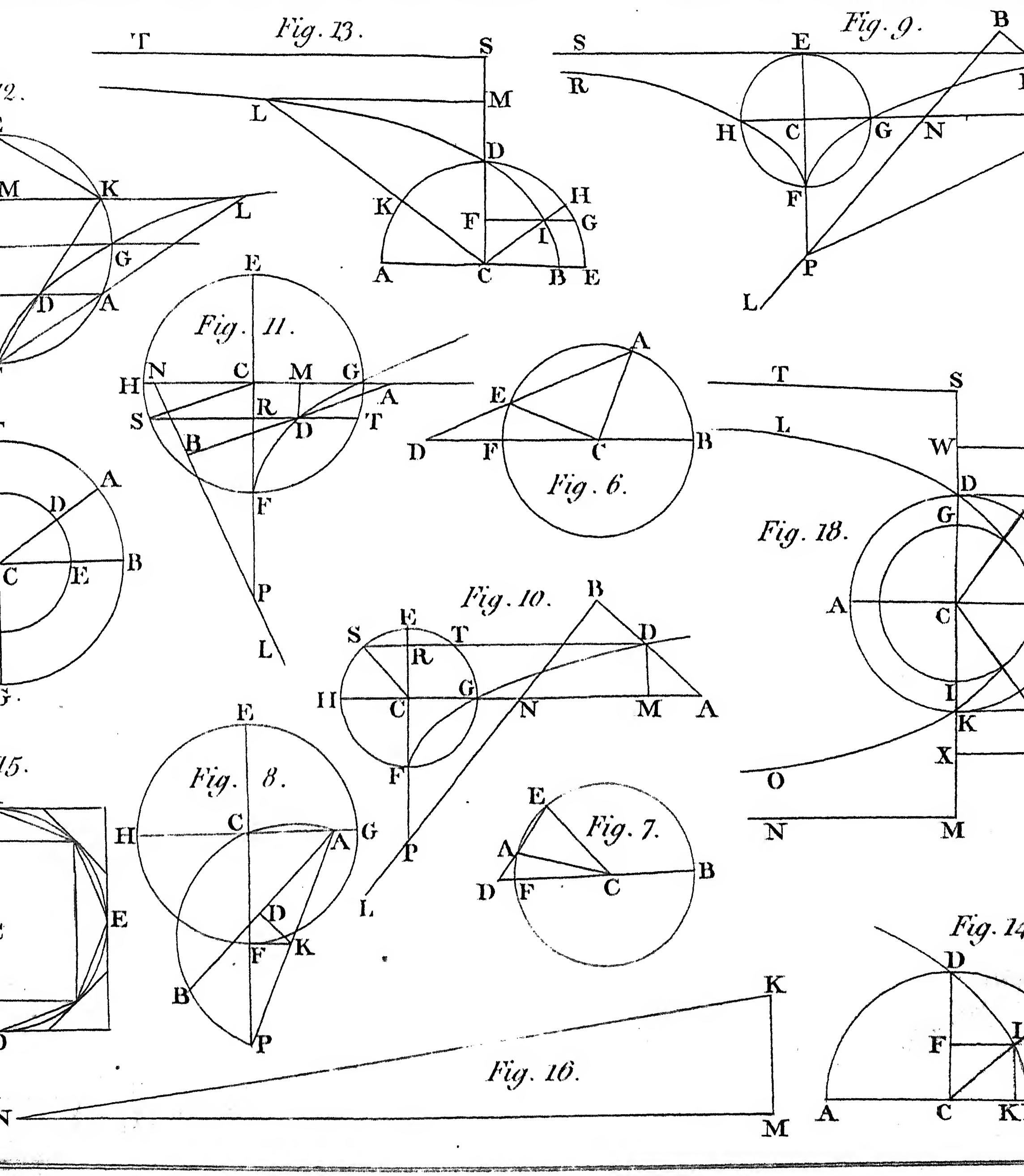
V.

The point V is called the *Vertex* of the cycloid.

VI.

A ſtraight line drawn from any point in the curve per-

Plate XV



perdendicular to the axis is called an *Ordinate* to the axis; and the ſegment of the axis between the vertex and an ordinate is called an *Abſciſs*. SECT. IV.

PROP. I.

During the revolution of the generating circle its center deſcribes a ſtraight line equal to the baſe of the cycloid; and the axis of the cycloid is equal to the diameter of the generating circle

For, the reſt being as in the Definitions, let C be the center of the generating circle, and A E the diameter paſſing through A. Then as the circle A W E always touches the ſtraight line A B, the center C is at the ſame diſtance from A B throughout the revolution, and therefore it muſt deſcribe a ſtraight line. At the commencement of the revolution, and alſo at the end of it, A C (18. iii.) is perpendicular to A B, and therefore it is evident that if theſe perpendiculars and the line deſcribed by the center were drawn, a parallelogram would be formed, of which the line deſcribed by the center and A B would be oppoſite ſides. The ſtraight line deſcribed by the center is therefore equal to the baſe of the cycloid. Laſtly, let A W E be the poſition of the generating circle at the beginning of the revolution, and then it is evident, from the firſt five Definitions, that at the middle of the revolution the point E will coincide with H and the deſcribing point A with V. The axis H V therefore of the cycloid is equal to the diameter of the generating circle. Fig. 20.

Cor. A circle deſcribed on the axis of a cycloid is equal to the generating circle.

PROP.

SECT. IV.

PROP. II.

An ordinate drawn from any point in the cycloid to the axis is equal to the arch of the generating circle, described on the axis, between the vertex and ordinate, together with the sine of the same arch.

Fig. 20. From any point F in the cycloid A V F B let F M be drawn an ordinate to H V the axis, and let it cut the circle V P H described on the axis in the point P; the ordinate F M is equal to the arch V P, between V the vertex and the ordinate, together with P M the sine of the same arch.

For let S F L I represent the generating circle when the describing point coincides with F, and in this situation let T denote its center, and let L be the point in which it touches A B. Draw the diameters L T S, F T I, and let L T S meet F M in N. Then S L (18. iii.) is perpendicular to A B; and, as F M is perpendicular to V H, M L is a parallelogram, and therefore (34. i.) M H is equal to N L, and M N is equal to H L. But, from the Cor. to the third Definition, the semicircle I L F is equal to H B half the base, and from the description of the curve the circular arch L F is equal to the remaining part L B of the base. Consequently H L is equal to the arch I L, or to its equal (26. iii.) S F; and as H M, L N are equal, it is evident (from Cor. 8. vi.) that M P is equal to N F, and the arch V P equal to the arch S F. The arch V P is therefore equal to M N; and as N F is equal to P M, the ordinate F M is equal to the arch V P together with P M the sine of the same arch.

Cor. The equation of the cycloid is immediately obtained from the above. For put F M $= y$, the arch V P $= z$, and P M $= s$, and then $z + s = y$, the equation of the curve.

DE-

DEFINITIONS.

SECT. IV.

VII.

Let A W E be a circle in which C is the center, A E a fixed diameter, and D a point in A E or in A E produced; and let the circle A W E roll along the ſtraight line A B ſo that every part of the circumference may touch it in regular ſucceſſion from the beginning of the revolution at A to the end of it at B; the curve D V F G deſcribed by the point D during the revolution is called a *Curtate Cycloid* if the point D be without the circle as in Fig. 21. but if D be within the circle as in Fig. 24. the curve D V F G is called a *Prolate or Inflected Cycloid*. Fig. 21. 24.

VIII.

The circle A W E is called the *Generating Circle* either of the Curtate or Prolate Cycloid.

IX.

The ſtraight line D G joining the points in which the generation of the curve begins and ends is called the *Baſe* of either of the two Cycloids.

X.

The ſtraight line R V biſecting the baſe D G at right angles and meeting the curve in V is called the *Axis*, and the point V is called the *Vertex* of either of the two cycloids.

XI.

A ſtraight line drawn from any point in the curve perpendicular to the axis is called an *Ordinate* to the axis; and the ſegment of the axis between the vertex and an ordinate is called an *Abſciſs*.

PROP. III.

In either the curtate or prolate cycloid the baſe is equal to the circumference of the generating circle; and in the curtate

SECT. IV.

curtate cycloid the axis is greater, but in the prolate it is lefs, than the diameter of the generating circle.

Fig. 21. 24.

For the reft remaining as in the Definitions, the generating circle both at the beginning and end of the revolution touches the ftraight line A B; and therefore D A B (18. iii.) is a right angle. And, as at the end of the revolution A D is reprefented by B G, B G is equal to A D, and A B G is a right angle. Confequently (28. i.) A D, B G are parallel, and therefore (33. i) A B, D G are equal and parallel. But as the revolution of the generating circle begins at A and ends at B, the circumference of the circle is equal to A B, and therefore the bafe D G is equal to the circumference of the generating circle. Again, as R V bifects D G at right angles, R H is parallel to D A or G B, and therefore A B is bifected in H, and A H is equal to half the circumference of the generating circle. Confequently in the middle of the revolution the point E will coincide with H, and the diameter E A as to pofition will coincide with the axis R V. Hence it is evident, that in the curtate cycloid the axis R V is greater than the diameter of the generating circle by the double of A D; but in the prolate cycloid the axis R V is lefs than the diameter of the generating circle by the double of A D.

Cor. From the above it is evident that in the middle of the revolution the center of the generating circle bifects the axis R V; and it is alfo evident that during the revolution the center of the generating circle is at the fame diftance from the bafe D G.

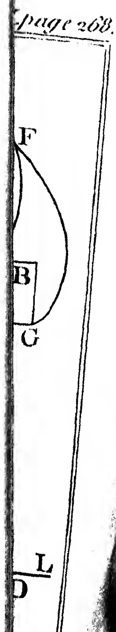
PROP. IV.

If an ordinate be drawn from any point in the curtate or prolate

prolate cycloid to the axis and cut a circle described on the axis as a diameter, the arch of the circle between the vertex and ordinate will be to the segment of the ordinate between the cycloid and circle as the circumference of the circle to the base of the cycloid. SECT. IV.

Let D V G be a curtate or prolate cycloid, of which D G is the base, R V the axis, and V the vertex, and from any point F in the cycloid let F M be drawn an ordinate to the axis, and let it cut the circle V P R, described upon V R as a diameter, in P; the arch V P is to the segment F P as the circumference of the circle V P R to the base D G. Fig. 21. 24.

For, the rest remaining as in the Definitions, let T be the center of the generating circle when the describing point D coincides with F. Through T draw the straight line S L Q perpendicular to D G, and let it meet A B in L. Then, as A B, D G are parallel, T L is perpendicular to A B; and, by the Cor. to the preceding Prop. and the description of the curve, T L is equal to the radius of the generating circle, and T Q is equal to the radius of the circle V P R. Through the points F and T draw the straight line F K I. With T as a center and T Q as a radius let the circle S F Q K be described; and with the same center and T L as a radius let the arch L I be described. Then, for the same reasons as in the demonstration of the second Proposition, M L is a parallelogram, P F is equal to H L, and the arch V P is equal to the arch S F. But the arch S F (26. iii.) is equal to the arch Q K; and as T L is equal to C A, from the generation of either cycloid, the arch L I is equal to the straight line H L. Again, as T Q K, T L I are similar sectors, by Cor. 2. and 3. to the Lemma in the preceding Section, (and 11. v.) the arch K Q : the arch L I : : circumference of the circle S F Q K : the



SECT. IV.

the circumference of the circle L I. Confequently, on account of the equals, the arch V P : the fegment F P : : the circumference of the circle V P R : the bafe D G.

Cor. The equation of the curtate cycloid, and alfo that of the prolate cycloid, is obtained from the above. For put $a =$ the circumference of the circle V P R, $b =$ the bafe D G, $z =$ the arch V P, $s =$ P M, and $y =$ F M; and then $a : b : : z : \frac{bz}{a} =$ P F. Confequently $y = \frac{bz}{a} + s$, the equation of the curtate and alfo that of the prolate cycloid.

SECTION V.

Of the Logarithmic Curve.

DEFINITIONS.

I.

If in the ftraight line X Y, of an unlimited length, fegments A B, B C, C D &c. be taken equal to one another, but indefinitely fmall, and if from the points of fection perpendiculars A E, B F, C G, D H &c. be drawn, and be in geometrical progreffion; the perpendiculars will be indefinitely near to one another, and the line drawn through their extremities E, F, G, H, &c. is called the *Logarithmic Curve.* Fig. 22.

II.

The ftraight line X Y is called the *Axis* to the logarithmic curve, and the perpendiculars A E, B F, C G, D H, &c. are called *Ordinates* to it.

PROPOSITION.

The axis is an afymptote to the logarithmic curve.

For, the reft remaining as in the Definitions, let the ordinates B F, C G, D H, &c. on the right of A E continually increafe; and, the fegments A *b*, *b c*, *c d*, &c. in the axis being equal to one another, and each equal to A B, let the ordinates *b f*, *c g*, *d h*, &c. on the left of A E continually decreafe; that is, let B F be to A E as A E to *b f*, and let A E, *b f*, *c g*, *d h*, &c. be in geometrical progreffion. Then ending the firft rank, and beginning the fecond with ordinates equally diftant from A E, we have the two following ranks of magnitudes proportional taken two and two in the fame order. Fig. 22.

SECT. V.

AE : BF : CG : DH

$db : cg : bf$: AE,

and therefore (22. V.) AE : DH :: db : AE; or DH : AE :: AE : db. Hence it is evident that the rectangle under any two ordinates equally diftant from AE is equal to the fquare of AE; and therefore if an ordinate on the right of AE be indefinitely great, an ordinate on the left of AE, and equally diftant from it, will be indefinitely fmall, but it can never become equal to nothing, or vanifh. Confequently the axis XY is an afymptote.

SCHOLIUM.

As AB, AC, AD, &c. conftitute a feries in arithmetical progreffion, and AE, BF, CG, DH, &c. a correfponding feries in geometrical progreffion, the fegments AB, AC, AD, &c. are analogous to a feries of natural numbers, and the ordinates BF, CG, DH, &c. are analogous to the logarithms of thefe numbers. The curve is named from thefe analogies.

The equation of the curve is deduced from the firft Definition, in the following manner. Put AE $= 1$, BF $= a$, and then $1 : a :: a : a^2 =$ CG; and $a : a^2 :: a^2 : a^3 =$ DH, &c. Hence it is evident that if x denote any number of equal parts AB, BC, CD in the axis, then will a^x be equal to the ordinate drawn through the extremity of the fegment in the axis denoted by x; and therefore if this ordinate be put equal to y, then $a^x = y$, which is the equation of the curve.

The logarithmic curve, on account of its equation, is alfo called an *Exponential Curve*.

OF

OF SPIRALS.

SECTION VI.

Of the Logarithmic Spiral.

DEFINITIONS.

I.

f any number of ſtraight lines CA, CB, CD, CE, &c. wn from the point C within the curve A B D E con- ı equal angles with it, the curve is called the *Loga- 'mic Spiral.* Fig. 23.

II.

'he point C is called the *Center* of the Logarithmic al; and any ſtraight line drawn from the center to curve is called an *Ordinate.*

PROPOSITION.

' number of ordinates of the logarithmic curve are in eometrical progreſſion, if they contain equal angles at 'e center.

.et A B D E be a logarithmic curve, of which C is center and C A, C B, C D, C E, &c. ordinates, and let angles A C B, B C D, D C E, &c. be equal to one :her; the ordinates C A, C B, C D, C E, &c. are in netrical progreſſion. Fig. 23.

or let the angles A C B, B C D, D C E, &c. be indefi- ly ſmall, and then the portions A B, B D, D E, &c. ıe curve may be conſidered as ſtraight lines, and as,

 by

SECT. VI.

by the firſt Definition, the angles CAB, CBD, CDE, &c. are equal, the triangles ACB, BCD, DCE, &c. are equiangular. Conſequently (4. vi.) AC : BC : : BC : DC; and BC : DC : : DC : EC. The Propoſition is therefore evident.

Cor. 1. If the ordinates BC, DC, EC, &c. on the right of AC ſucceſſively increaſe, the ordinates *b*C, *d*C, *e*C, &c. on the left of AC will ſucceſſively decreaſe. For the reſt remaining as above, if the angles ACB, AC*b*, *b*C*d*, *d*C*e*, be equal to one another, it may be proved as above that BC : AC : : AC : *b*C, and AC : *b*C : : *b*C : *d*C, &c.

Cor. 2. It may be proved, as in the logarithmic curve, that AC is a mean proportional between ordinates equally diſtant from it; and therefore as the logarithmic curve cannot meet its aſymptote at any aſſigned diſtance, ſo the logarithmic ſpiral cannot fall into its center at any aſſigned number of revolutions.

SCHOLIUM.

As the angles ACB, ACD, ACE, &c. conſtitute a ſeries in arithmetical progreſſion, and the ordinates AC, BC, DC, EC, &c. a ſeries in geometrical progreſſion, the ordinates are analogous to natural numbers, and the angles to their logarithms; and from this conſideration the ſpiral receives its name.

If AC be put $= 1$, BC $= a$, $x =$ any angle in the arithmetical ſeries, and $y =$ the correſponding ordinate, then $a^x = y$, the equation of the logarithmic ſpiral, for the ſame reaſons as ſtated in the ſcholium on the logarithmic curve.

SECTION

SECTION VII.

Of the Spiral of Archimedes.

DEFINITIONS.

I.

If the ſtraight line C L revolve in a plane about one of its extremities C as a center, with an uniform velocity, and if at the commencement of the revolution a point begin to move from C and proceed in C L towards L with an uniform velocity, the line deſcribed by the point moving in C L is called *the Spiral of Archimedes*. Fig. 25.

II.

The line C G A deſcribed in the firſt revolution of C L is called *the Firſt Spiral*; the line A H B deſcribed in the ſecond revolution of C L is called *the Second Spiral*; the line B K D deſcribed in the third revolution of C L is called *the Third Spiral*, &c.

III.

If A be a fixed point in C L, and C A be the diſtance moved over by the deſcribing point during the firſt revolution, the circle deſcribed by the point A during the revolution is called *the Generating Circle*; and the point C is called the *Center* of any one of the ſpirals.

IV.

A ſtraight line drawn from C to the point in which any one of the ſpirals ends is called *the Axis* of that ſpiral; and a ſtraight line drawn from C to any other point in the ſpiral is called *an Ordinate*.

 PROP.

SECT. VII.

PROP. I.

If an ordinate be drawn to any point in the first spiral, and be produced till it cut the generating circle, the ordinate will be a fourth proportional to the circumference of the generating circle, the axis, and the circular arch generated from the beginning of the revolution to the point of section.

Fig. 26. Let A G be an ordinate to the first spiral A G B, of which B K H Z is the generating circle, and A being the center, let A B be the axis, and let A G be produced and cut the circle in H; the ordinate A G is a fourth proportional to the circumference B K H Z, the axis A B, and the circular arch B K H, generated from the beginning of the revolution to H the point of section.

For the line in which the fixed point B is situated revolves about A with an uniform velocity, and the point describing the spiral moves from A towards B with an uniform velocity, according to the third and first Definitions. The circumference of the circle therefore and the axis A B are described by uniform velocities in the same time; and for the same reasons the arch B K H and the ordinate A G are described by the same uniform velocities in the same time. But spaces described by the same uniform velocity are to one another as the times in which they are described, and therefore the circumference of the circle B K H Z is to the arch B K H as the time of one revolution to the time of describing the arch B K H, and A B is to A G as the same portions of time are to one another. Consequently (11. V. and alternation) the circumference of the circle B K H Z is to the axis A B as the arch B K H to the ordinate A G.

Cor. 1. The rest remaining, if A F another ordinate be produced and cut the generating circle in Z, by the above

Cor. 2. If the axis of the first spiral, or, which is the same thing, the radius of the generating circle, be put equal to r, the circumference of the generating circle equal to c, any arch B K H from the axis equal to z, and the corresponding ordinate A G equal to y, then $c : r :: z : \frac{rz}{c} = y$, the equation of the first spiral.

PROP. II.

An ordinate drawn to any point in the second spiral is a fourth proportional to the circumference of the generating circle, the radius of the circle, and the sum of the circumference of the circle and its arch generated from the beginning of the revolution to the ordinate.

Let A E be an ordinate drawn to any point E in the second spiral B E D M, and let it cut the generating circle B K H Z in H. Let A be the center and A M the axis, and consequently the situation from which the revolution begins. The circumference B K H Z of the generating circle is to its radius A B as the sum of the circumference B K H Z and the arch B K H to the ordinate A E. Fig. 26.

For, as A L revolves with an uniform velocity, the circumference B K H Z is to the sum of the circumference B K H Z and the arch B K H as the time of one revolution to the time in which the circumference and the arch B K H are generated by B the extremity of the radius. And these portions of time are to one another as the radius A B to the ordinate A E, as the point moving in A L with an uniform velocity passes over A B, A E in these portions of time respectively. Consequently (11. v. and alternation) the circumference B K H Z :

the

SECT. VII. the radius A B : : the circumference B K H Z + the arch B K H : the ordinate A E.

Cor. 1. The reſt remaining as above, if A D another ordinate to the ſpiral B E D M cut the generating circle in Z, then, by the above, (and 11. v.) the circumference B K H Z + the arch B K H : A E : : the circumference B K H Z + the arch B K Z : A D.

Cor. 2. Put $c =$ the circumference of the generating circle, $r =$ its radius, $y =$ the ordinate A E, and $z =$ the arch B K H; and then, by the Propoſition, $c + z : y :: c : r$, and therefore the equation of the ſecond ſpiral is $r \times \overline{c+z} = cy$.

Cor. 3. If c denote the circumference of the generating circle, r its radius, y an ordinate drawn to any point in the nth ſpiral, and z the arch of the generating circle generated from the beginning of the revolution to the ordinate, it may be proved, as above, that $c : r :: \overline{n-1} \times c + z : y$. Conſequently $r \times \overline{nc - c + z} = cy$ is a general equation for any one of the ſpirals of Archimedes.

Cor. 4. The reſt of the notation remaining as above, let an ordinate v contain an angle with the ordinate y, and let d denote the arch in the generating circle which meaſures this angle. Let Z denote an arch of the generating circle, generated from the beginning of the revolution to the ordinate Y; and let the ordinate Y and the ordinate x contain an angle equal to the angle contained by y and v, and conſequently alſo meaſured by an arch d in the generating circle. Then by the laſt Cor. (and 11. v.) $\overline{n-1} \times c + z : y :: \overline{n-1} \times c + z + d : v :: \overline{n-1} \times c + Z : Y :: \overline{n-1} \times c + Z + d : x$; and therefore, by Lemma X. page 154. (and 11. v.) $d : v - y :: d : x - Y$. Conſequently (14. v.) the

the excefs of v above y is equal to the excefs of x above Y.

Cor. 5. From the laft Cor. it is evident, that if any number of ordinates contain angles which conftitute a feries in arithmetical progreffion, the ordinates themfelves will be in arithmetical progreffion.

SECTION VIII.

Of the Hyperbolic or Reciprocal Spiral.

DEFINITIONS.

I.

Fig. 27. If with C, one of the extremities of the ſtraight line C L, as a center and diſtances C A, C B, C D, &c. in C L, arches of circles A G, B H, D I, &c. be deſcribed, and if theſe arches be equal to one another, and indefinitely near; the curve G H I, &c. drawn through their extremities is called *the Hyperbolic or Reciprocal Spiral.*

II.

The ſtraight line C L is called the *Axis* of the hyperbolic ſpiral, the point C its *Center*, and any ſtraight line C K drawn from the center to the curve is called an *Ordinate.*

PROP. I.

In the hyperbolic ſpiral any two ordinates are reciprocally to one another as the angles they contain with the axis.

Fig. 28. Let E G I be a hyperbolic ſpiral, of which C is the center, C L the axis, and C I, C G any two ordinates; C I is to C G as the angle L C G to the angle L C I.

For let the circular arches G A, I D be deſcribed, and let them meet the axis in A and D. Let C G produced meet the arch D I produced in M; and then as D C M, A C G are ſimilar ſectors, by Cor. 3. to the Lemma in the third ſection, C D : C A : : the arch D M : the arch A G. But C I is equal to C D, and C G to C A,

and

and by the firft Definition the arch A G is equal to the arch D I; and therefore C I : C G : : the arch D M : the arch D I. Again (33. vi.) the arch D M : the arch D I : : the angle D C M or L C G : the angle D C I or L C I. Confequently (11. v.) C I is to C G as the angle L C G to the angle L C I. SECT. VIII.

Cor. Every thing remaining as above, let C I be a given ordinate, and put it equal to r; put the arch D I $= a$, C G $= y$, and the arch D M $= z$. Then, by the above, $r : y : : z : a$, and $ar = yz$, the equation of the curve.

PROP. II.

If from the center of an hyperbolic fpiral a ftraight line be drawn at right angles to the axis and equal to one of the circular arches intercepted between the curve and axis, a ftraight line drawn through the extremity of this perpendicular parallel to the axis will be an afymptote to the curve.

Let E G I be an hyperbolic fpiral, of which C is the center and C L the axis, and let the ftraight line C B at right angles to C L be equal to I D any circular arch intercepted between the curve and the axis; the ftraight line B F drawn through B parallel to C L is an afymptote to the curve. Fig. 28.

For through I draw N F parallel to C B, or perpendicular to C L, and let it meet B F in F; and then N F (34. i.) is equal to C B, and therefore equal to the arch D I. But the arch D I is greater than its fine N I, and therefore the point F in B F is without the curve. Again, from the preceding Propofition it is manifeft, that if the ordinate C I be indefinitely great when compared to another ordinate C G, the angle D C I is indefinitely fmall when compared to A C G; and when an

SECT. VIII.

angle is indefinitely ſmall, the exceſs of the arch which meaſures it above its ſine is leſs than any aſſigned magnitude. Conſequently as I F is the exceſs of the arch D I above its ſine N I, if the curve E G I be continued till the angle D C I become indefinitely ſmall, the diſtance I F of the curve from B F will be leſs than any aſſigned magnitude. The ſtraight line B F is therefore an aſymptote to the curve.

THE END.

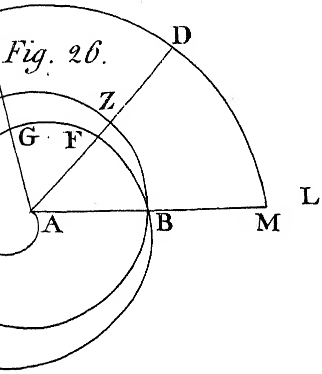


Fig. 26.

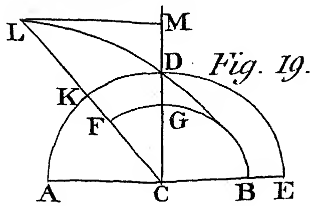
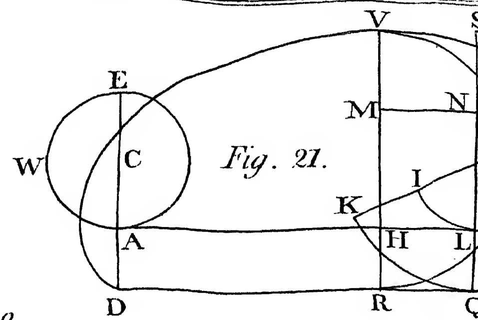


Fig. 19.



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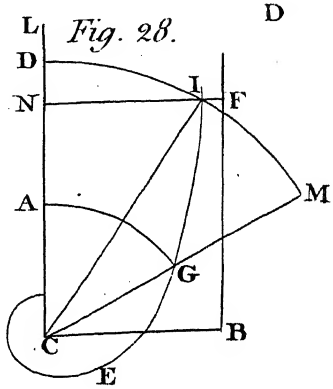

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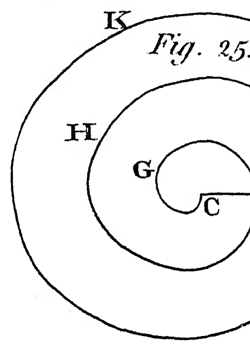

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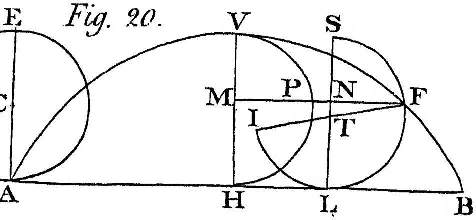

Fig. 20.

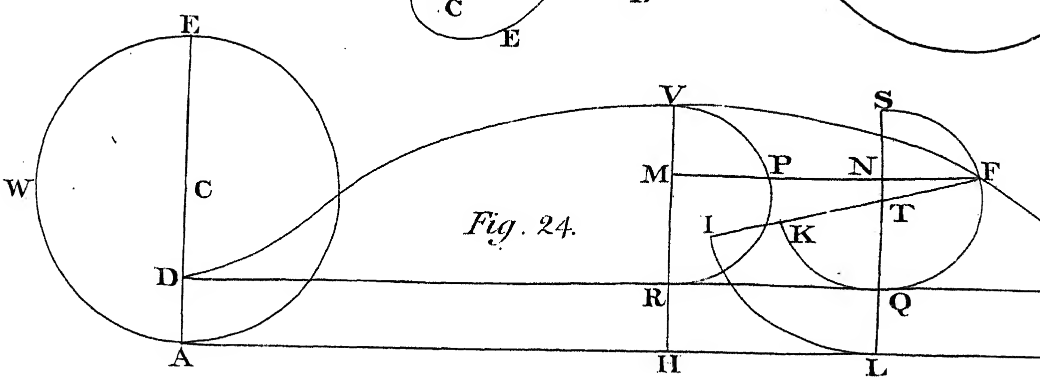

Fig. 24.

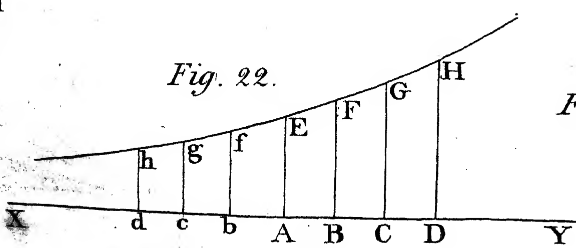

Fig. 22.

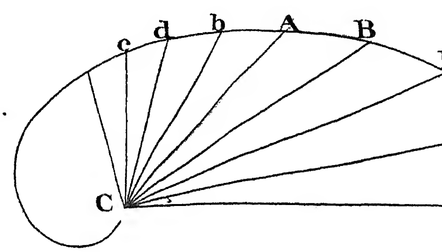

Fig. 23.